Solutions to Selected Exercises from Chapter 10
Bain & Engelhardt - Second Edition

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Exercise 1
The joint pdf of $X_1, \ldots, X_n$ is $f(x_1, \ldots, x_n; \mu) = \prod_{i=1}^n \left(\frac{e^{-\mu} \mu^{x_i}}{x_i!} \right) = \frac{e^{-n\mu} \mu^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}$. Moreover, since $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{POI}(\mu)$ we have $S = \sum_{i=1}^n X_i \sim \text{POI}(n\mu)$. The pdf of S is thus $f(s; \mu) = \frac{e^{-n\mu} n^s \mu^s}{s!}$. The conditional pdf

$$f_{\boldsymbol{X}|s} = \begin{cases} \frac{f(x_1,\ldots,x_n;\mu)}{f(s;\mu)} = \frac{\frac{e^{-n\mu} \mu^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}}{\frac{e^{-n\mu} n^s \mu^s}{s!}} = \frac{s!}{n^s \prod_{i=1}^n x_i!} & \text{if } \sum_{i=1}^n x_i = s \\ 0 & \text{otherwise,} \end{cases}$$

does not depend on μ, hence $S = \sum_{i=1}^n X_i$ is sufficient for μ.

Exercise 6
The joint pdf of $X_1, \ldots, X_n$ is

$$\begin{aligned} f(x_1, \ldots, x_n; p) &= \prod_{i=1}^n \binom{m_i}{x_i} p^{x_i} (1-p)^{m_i - x_i} = \left[\prod_{i=1}^n \binom{m_i}{x_i} \right] p^{\sum_{i=1}^n x_i} \frac{(1-p)^{\sum_{i=1}^n m_i}}{(1-p)^{\sum_{i=1}^n x_i}} \\ &= \underbrace{p^s \frac{(1-p)^{\sum_{i=1}^n m_i}}{(1-p)^s}}_{=g(s;p)} \underbrace{\left[\prod_{i=1}^n \binom{m_i}{x_i} \right]}_{=h(x_1,\ldots,x_n)}, \end{aligned}$$

where $s = \sum_{i=1}^n x_i$. By the factorization criterion, $S = \sum_{i=1}^n X_i$ is sufficient for p.

Exercise 11
We will use the factorization criterion to answer both subquestion. Note that the joint pdf of $X_1, \ldots, X_n$ is equal to

$$f(x_1, \ldots, x_n; \theta_1, \theta_2) = \prod_{i=1}^n \left(\frac{1}{\theta_2 - \theta_1} I_{(\theta_1, \theta_2)}(x_i) \right) = \frac{1}{(\theta_2 - \theta_1)^n} I_{(\theta_1, \infty)}(x_{1:n}) I_{(-\infty, \theta_2)}(x_{n:n}).$$

(a) If θ_2 is known, then θ_1 is the only parameter to consider. We write

$$f(x_1, \ldots, x_n; \theta_1) = \underbrace{\frac{1}{(\theta_2 - \theta_1)^n} I_{(\theta_1, \infty)}(s)}_{=g(s;\theta_1)} \underbrace{I_{(-\infty, \theta_2)}(x_{n:n})}_{=h(x_1,\ldots,x_n)}$$

with $s = x_{1:n}$. By the factorization criterion, $S = X_{1:n}$ is sufficient for θ_1.

(b) We now treat both θ_1 and θ_2 as unknown parameters. The required factorization is now

$$f(x_1, \ldots, x_n; \theta_1, \theta_2) = \underbrace{\frac{1}{(\theta_2 - \theta_1)^n} I_{(\theta_1,\infty)}(s_1) I_{(-\infty,\theta_2)}(s_2)}_{=g(s_1,s_2;\theta_1,\theta_2)} \times \underbrace{1}_{=h(x_1,\ldots,x_n)} .$$

$S_1 = X_{1:n}$ and $S_2 = X_{n:n}$ are jointly sufficient for θ_1 and θ_2.

Exercise 13

The joint pdf of $X_1, \ldots, X_n$ can be written as

$$\begin{aligned} f(x_1, \ldots, x_n; \theta_1, \theta_2) &= \prod_{i=1}^{n} \left(\frac{\Gamma(\theta_1 + \theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)} x_i^{\theta_1 - 1} (1 - x_i)^{\theta_2 - 1} \right) \\ &= \left(\frac{\Gamma(\theta_1 + \theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)} \right)^n \left(\prod_{i=1}^{n} x_i \right)^{\theta_1 - 1} \left(\prod_{i=1}^{n} (1 - x_i) \right)^{\theta_2 - 1} \\ &= \underbrace{\left(\frac{\Gamma(\theta_1 + \theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)} \right)^n s_1^{\theta_1 - 1} s_2^{\theta_2 - 1}}_{=g(s_1,s_2;\theta_1,\theta_2)} \times \underbrace{1}_{=h(x_1,\ldots,x_n)} , \end{aligned}$$

where we defined $s_1 = \prod_{i=1}^{n} x_i$ and $s_2 = \prod_{i=1}^{n} (1 - x_i)$. According to the factorization crition, $S_1 = \prod_{i=1}^{n} X_i$ and $S_2 = \prod_{i=1}^{n} (1 - X_i)$ are jointly sufficient for θ_1 and θ_2.

Exercise 19

The pdf depends on $k = 1$ unknown parameter, namely μ. However, if we expand the square in the exponential, that is

$$f(x; \mu) = \frac{1}{\sqrt{2\pi}|\mu|} e^{-\frac{(x-\mu)^2}{2\mu^2}} = \frac{1}{\sqrt{2\pi}|\mu|} e^{-\frac{x^2 - 2\mu x + \mu^2}{2\mu^2}} = \frac{1}{\sqrt{2\pi}|\mu|} e^{-\left(\frac{x^2}{2\mu^2} - \frac{x}{\mu} + \frac{1}{2}\right)} = \frac{e^{-\frac{1}{2}}}{\sqrt{2\pi}|\mu|} e^{-\frac{x^2}{2\mu^2} + \frac{x}{\mu}},$$

then we see that the exponential contains two summands of the form $q_j(\mu) t_j(x)$. The $\mathrm{N}(\mu, \mu^2)$ is thus not a member of the REC.

Exercise 20

(a) The pdf can be written as

$$f(x; p) = p^x (1 - p)^{1 - x} = (1 - p) \left(\frac{p}{1 - p} \right)^x = (1 - p) e^{x \ln\left(\frac{p}{1-p}\right)},$$

such that it is a member of the REC with $c(p) = 1 - p$, $h(x) = 1$, $q_1(p) = \ln\left(\frac{p}{1-p}\right)$, and $t_1(x) = x$. Hence $S = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic for p.

(b) The pdf can be written as

$$f(x; \mu) = \frac{e^{-\mu} \mu^x}{x!} = \frac{1}{x!} e^{-\mu} e^{x \ln \mu},$$

such that it is a member of the REC with $c(\mu) = e^{-\mu}$, $h(x) = \frac{1}{x!}$, $q_1(\mu) = \ln \mu$ and $t_1(x) = x$. Hence $S = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic for μ.

(c) The pdf can be written as

$$f(x;p) = \binom{x-1}{r-1} p^r (1-p)^{x-r} = \binom{x-1}{r-1} \left(\frac{p}{1-p}\right)^r (1-p)^x$$
$$= \binom{x-1}{r-1} \left(\frac{p}{1-p}\right)^r e^{x\ln(1-p)},$$

such that it is a member of the REC with $c(p) = \left(\frac{p}{1-p}\right)^r$, $h(x) = \binom{x-1}{r-1}$, $q_1(p) = \ln(1-p)$, and $t_1(x) = x$. Hence $S = \sum_{i=1}^n X_i$ is a complete sufficient statistic for p.

(d) The pdf can be written as

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2-2\mu x+\mu^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}+\frac{\mu x}{\sigma^2}-\frac{\mu^2}{2\sigma^2}}$$
$$= \frac{e^{-\frac{\mu^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} e^{\frac{\mu}{\sigma^2}x - \frac{1}{2\sigma^2}x^2},$$

such that it is a member of the REC with $c(\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\mu^2}{2\sigma^2}}$, $h(x) = 1$, $q_1(\mu,\sigma^2) = \frac{\mu}{\sigma^2}$, $q_2(\mu,\sigma^2) = -\frac{1}{2\sigma^2}$, $t_1(x) = x$, and $t_2(x) = x^2$. Hence $S_1 = \sum_{i=1}^n X_i$ and $S_2 = \sum_{i=1}^n X_i^2$ are jointly complete sufficient statistics for μ and σ^2.

(e) The pdf can be written as

$$f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}} = \frac{1}{\theta} e^{-\frac{1}{\theta}x},$$

such that it is a member of the REC with $c(\theta) = \frac{1}{\theta}$, $h(x) = 1$, $q_1(\theta) = -\frac{1}{\theta}$, and $t_1(x) = x$. Hence $S = \sum_{i=1}^n X_i$ is a complete sufficient statistic for θ.

(f) The pdf can be written as

$$f(x;\theta,\kappa) = \frac{1}{\theta^\kappa \Gamma(\kappa)} x^{\kappa-1} e^{-\frac{x}{\theta}} = \frac{1}{\theta^\kappa \Gamma(\kappa)} e^{(\kappa-1)\ln(x)} e^{-\frac{x}{\theta}} = \frac{1}{\theta^\kappa \Gamma(\kappa)} e^{-\frac{1}{\theta}x + (\kappa-1)\ln(x)},$$

such that it is a member of the REC with $c(\theta,\kappa) = \frac{1}{\theta^\kappa\Gamma(\kappa)}$, $h(x) = 1$, $q_1(\theta,\kappa) = -\frac{1}{\theta}$, $q_2(\theta,\kappa) = \kappa - 1$, $t_1(x) = x$, and $t_2(x) = \ln(x)$. Hence $S_1 = \sum_{i=1}^n X_i$ and $S_2 = \sum_{i=1}^n \ln(X_i)$ are jointly complete sufficient statistics for θ and κ.

(g) The pdf can be written as

$$f(x;\theta_1,\theta_2) = \frac{\Gamma(\theta_1+\theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)} x^{\theta_1-1}(1-x)^{\theta_2-1} = \frac{\Gamma(\theta_1+\theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)} e^{(\theta_1-1)\ln(x)} e^{(\theta_2-1)\ln(1-x)}$$
$$= \frac{\Gamma(\theta_1+\theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)} e^{(\theta_1-1)\ln(x)+(\theta_2-1)\ln(1-x)},$$

such that it is a member of the REC with $c(\theta_1,\theta_2) = \frac{\Gamma(\theta_1+\theta_2)}{\Gamma(\theta_1)\Gamma(\theta_2)}$, $h(x) = 1$, $q_1(\theta_1,\theta_2) = \theta_1 - 1$, $q_2(\theta_1,\theta_2) = \theta_2 - 1$, $t_1(x) = \ln(x)$, and $t_2(x) = \ln(1-x)$. Hence $S_1 = \sum_{i=1}^n \ln(X_i)$ and $S_2 = \sum_{i=1}^n \ln(1-X_i)$ are jointly complete sufficient statistics for θ_1 and θ_2.

(h) Note that β is considered to be known. The pdf can be written as

$$f(x;\theta) = \frac{\beta}{\theta^\beta} x^{\beta-1} e^{-\left(\frac{x}{\theta}\right)^\beta} = \frac{\beta}{\theta^\beta} x^{\beta-1} e^{-\frac{1}{\theta^\beta}x^\beta},$$

such that it is a member of the REC with $c(\theta) = \frac{\beta}{\theta^\beta}$, $h(x) = x^{\beta-1}$, $q_1(\theta) = -\frac{1}{\theta^\beta}$, and $t_1(x) = x^\beta$. Hence $S = \sum_{i=1}^n X_i^\beta$ is a complete sufficient statistic for θ.

Exercise 21
In part (a) and (b) we are asked to find UMVUEs. The approach is as follows. From Exercise 20(a) we know that $S = \sum_{i=1}^n X_i$ is a complete sufficient statistic for p. It is also easy to show that the MLE for p is equal to $\hat{p} = \bar{X} = \frac{1}{n}S$. We will therefore make an educated guess for the estimator. If this proposed estimator is unbiased, then we have immediately found an UMVUE. If this approach leads to a biased estimator, then we try a transformation to remove the bias. Also note that $S = \sum_{i=1}^n X_i \sim \mathrm{BIN}(n,p)$ such that $\mathbb{E}(S) = np$ and $\mathbb{V}\mathrm{ar}(S) = np(1-p)$.

(a) We try the estimator $\hat{p}(1-\hat{p}) = \frac{S}{n}\left(1 - \frac{S}{n}\right)$. Its expectation is

$$\begin{aligned}\mathbb{E}\left(\hat{p}(1-\hat{p})\right) &= \mathbb{E}\left(\frac{S}{n}\left(1-\frac{S}{n}\right)\right) = \frac{\mathbb{E}(S)}{n} - \frac{\mathbb{E}(S^2)}{n^2} = p - \frac{\mathbb{V}\mathrm{ar}(S) + (\mathbb{E}(S))^2}{n^2} \\ &= p - \frac{np(1-p) + (np)^2}{n^2} = \frac{np - p(1-p) - np^2}{n} = \frac{np(1-p) - p(1-p)}{n} \\ &= \frac{n-1}{n}p(1-p).\end{aligned}$$

Hence $T = \frac{n}{n-1}\left[\frac{S}{n}\left(1-\frac{S}{n}\right)\right] = \frac{S}{n-1}\left(1-\frac{S}{n}\right)$ is unbiased for $p(1-p)$ such that it is also an UMVUE.

(b) Note that $p^2 = p - p(1-p)$ is a linear combination of the terms p and $p(1-p)$. The unbiased estimators for both parts are $\frac{S}{n}$ and $\frac{S}{n-1}\left(1-\frac{S}{n}\right)$ (see previous part), respectively. We will therefore try $\frac{S}{n} - \frac{S}{n-1}\left(1-\frac{S}{n}\right)$. Linear of the expectation gives

$$\mathbb{E}\left(\frac{S}{n} - \frac{S}{n-1}\left(1-\frac{S}{n}\right)\right) = \mathbb{E}\left(\frac{S}{n}\right) - \mathbb{E}\left(\frac{S}{n-1}\left(1-\frac{S}{n}\right)\right) = p - p(1-p) = p^2.$$

Hence $T = \frac{S}{n} - \frac{S}{n-1}(1-\frac{S}{n}) = \frac{S(S-1)}{n(n-1)}$ is unbiased for p^2 such that it is also an UMVUE.

Exercise 22
We have seen in Exercise 20(b), that $S = \sum_{i=1}^n X_i$ is a complete sufficient statistic for μ. According to Lehmann-Scheffé, Theorem 10.4.1 on page 346 of B&E, we can find an UMVUE if we can find an unbiased estimator for $e^{-\mu}$ that is a function of S only. In exercise 33(g) we have seen that $\mathbb{E}\left((\frac{n-1}{n})^S\right) = e^{-\mu}$. $\left(\frac{n-1}{n}\right)^{\sum_{i=1}^n X_i}$ is thus an UMVUE for $e^{-\mu}$.

Exercise 25
The pdf can be written as

$$f(x;\theta) = \theta x^{\theta-1} = \theta e^{(\theta-1)\ln(x)} = \theta e^{(1-\theta)(-\ln x)},$$

such that it is a member of the REC with $c(\theta) = \theta$, $h(x) = 1$, $q_1(\theta) = 1-\theta$, and $t_1(x) = -\ln(x)$. Hence $S = -\sum_{i=1}^n \ln(X_i)$ is a complete sufficient statistic for θ.

(a) Using the hint we find $\mathbb{E}(S) = \sum_{i=1}^n \mathbb{E}\left(-\ln(X_i)\right) = \frac{n}{\theta}$. We conclude that $-\frac{1}{n}\sum_{i=1}^n \ln(X_i)$ is an unbiased estimator for $\frac{1}{\theta}$ and only a function of S. According to Lehmann-Scheffé, Theorem 10.4.1 on page 346 of B&E, this is also an UMVUE.

(b) Having found $\frac{S}{n} = -\frac{1}{n}\sum_{i=1}^{n}\ln(X_i)$ as an UMVUE for $\frac{1}{\theta}$, we might try $\frac{n}{S}$ as an UMVUE for θ. This estimator is still a function of S only but it is no longer clear that it is unbiased. We have to compute $\mathbb{E}\left(\frac{n}{S}\right)$. If X has the pdf $f(x;\theta)$, then $Y = -\ln(X)$ has the pdf

$$f_Y(y;\theta) = f_X(e^{-y};\theta)\ \ |-e^{-y}| = \theta(e^{-y})^{(\theta-1)}e^{-y} = \theta e^{-\theta y}, \qquad y > 0.$$

From Table B.2 we can see that $Y \sim \text{GAM}\big(\frac{1}{\theta},1\big)$. Using the properties of MGFs we also find $S = \sum_{i=1}^{n} Y_i = -\sum_{i=1}^{n}\ln(X_i) \sim \text{GAM}\big(\frac{1}{\theta},n\big)$. The pdf of S is thus

$$f_S(s) = \frac{1}{\left(\frac{1}{\theta}\right)^n\Gamma(n)}s^{n-1}e^{-\theta s}, \qquad s > 0,$$

and we can compute

$$\begin{aligned}
\mathbb{E}\left(\frac{n}{S}\right) &= \int_0^\infty \frac{n}{s}\frac{1}{\left(\frac{1}{\theta}\right)^n\Gamma(n)}s^{n-1}e^{-\theta s}ds = n\int_0^\infty \left(\frac{\Gamma(n-1)}{\Gamma(n-1)}\right)\frac{1}{\left(\frac{1}{\theta}\right)^n\Gamma(n)}s^{(n-1)-1}e^{-\theta s}ds \\
&= n\theta\frac{\Gamma(n-1)}{\Gamma(n)}\int_0^\infty \underbrace{\frac{1}{\left(\frac{1}{\theta}\right)^{n-1}\Gamma(n-1)}s^{(n-1)-1}e^{-\theta s}}_{\text{pdf of GAM}(\frac{1}{\theta},n-1)}\,ds \\
&= n\theta\frac{\Gamma(n-1)}{\Gamma(n)} = \frac{n}{n-1}\theta,
\end{aligned}$$

where we have used the fact that the pdf of the GAM$(\frac{1}{\theta},n-1)$ integrated over its support should be equal to 1. This calculations suggest that

$$\hat{\theta} = \frac{n-1}{n}\frac{n}{S} = \frac{n-1}{S} = -\frac{n-1}{\sum_{i=1}^{n}\ln(X_i)},$$

is an estimator for θ which is (1) unbiased, and (2) a function of S only. According to Lehmann-Scheffé, Theorem 10.4.1 on page 346 of B&E, this is also an UMVUE.

Note 1: It also possible to solve the integral directly.

Note 2: One could have seen immediately from Jensen's inequality that $\frac{n}{S}$ will give a biased estimator for θ. It was thus clear from the start that a correction was necessary.

Exercise 31

(a) The likelihood and log-likelihood are $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \theta^n\left(\prod_{i=1}^{n}(1+x_i)\right)^{-(1+\theta)}$ and $\ln L(\theta) = n\ln(\theta) - (1+\theta)\sum_{i=1}^{n}\ln(1+x_i)$, respectively. The first and second derivative of the log-likelihood are:

$$\begin{aligned}
\frac{d}{d\theta}\ln L(\theta) &= \frac{n}{\theta} - \sum_{i=1}^{n}\ln(1+x_i) \\
\frac{d^2}{d\theta^2}\ln L(\theta) &= -\frac{n}{\theta^2} < 0, \qquad \text{for all } \theta.
\end{aligned}$$

Because the second derivative is always negative, we find the ML estimator as follows:

$$\frac{n}{\hat{\theta}} - \sum_{i=1}^{n}\ln(1+X_i) = 0 \qquad \Rightarrow \qquad \hat{\theta} = \frac{n}{\sum_{i=1}^{n}\ln(1+X_i)}.$$

(b) The pdf can be written as

$$f(x;\theta) = \theta(1+x)^{-(1+\theta)} = \theta e^{-(1+\theta)\ln(1+x)},$$

such that it is a member of the REC with $c(\theta) = \theta$, $h(x) = 1$, $q_1(\theta) = -(1+\theta)$, and $t_1(x) = \ln(1+x)$. Hence $S = \sum_{i=1}^n \ln(1+X_i)$ is a complete sufficient statistic for θ.

(c) For the numerator of the CRLB, $\tau(\theta) = \frac{1}{\theta}$ yields $\tau'(\theta) = -\frac{1}{\theta^2}$. The following results are helpful to find the denominator:

$$\frac{\partial^2}{\partial\theta^2} \ln f(x;\theta) = -\frac{1}{\theta^2},$$
$$\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2} \ln f(X;\theta)\right) = \mathbb{E}\left(-\frac{1}{\theta^2}\right) = -\frac{1}{\theta^2}.$$

Overall, the CRLB is

$$\frac{[\tau'(\theta)]^2}{-n\,\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2}, \ln f(X;\theta)\right)} = \frac{\frac{1}{\theta^4}}{\frac{n}{\theta^2}} = \frac{1}{n\theta^2}.$$

(d) It seems intuitive to use $\frac{1}{\hat{\theta}} = \frac{1}{n}\sum_{i=1}^n \ln(1+X_i)$ to estimate θ. This estimator is already a function of S but we do not know yet whether it is biased or not. If X has the pdf $f(x;\theta) = \theta(1+x)^{-(1+\theta)}$, then the pdf of random variable $Y = \ln(1+X)$ is given by

$$f_Y(y;\theta) = f_X(e^y - 1;\theta)\,|e^y| = \theta(e^y)^{-(1+\theta)}e^y = \theta e^{-\theta y}, \qquad y > 0.$$

Hence $Y \sim \mathrm{GAM}(\frac{1}{\theta}, 1)$ and in turn $S = \sum_{i=1}^n Y_i = \sum_{i=1}^n \ln(1+X_i) \sim \mathrm{GAM}(\frac{1}{\theta}, n)$ (see Example 6.4.6 in the book). Since $\mathbb{E}(S) = \frac{n}{\theta}$, the estimator $T = \frac{S}{n} = \frac{1}{n}\sum_{i=1}^n \ln(1+X_i)$ is unbiased for $\tau(\theta) = \frac{1}{\theta}$ such that it is also an UMVUE.

(e) The CRLB for θ is

$$\frac{1}{-n\,\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2} \ln f(X;\theta)\right)} = \frac{1}{\frac{n}{\theta^2}} = \frac{\theta^2}{n}$$

such that the asymptotic distribution of the MLE $\hat{\theta}_n$ is

$$\frac{\hat{\theta}_n - \theta}{\sqrt{CRLB}} = \frac{\hat{\theta}_n - \theta}{\theta/\sqrt{n}} \xrightarrow{d} Z \sim \mathrm{N}(0,1).$$

The CRLB for $1/\theta$ was derived in part (c). The asymptotic distribution of the MLE $\tau(\hat{\theta}_n) = \frac{1}{\hat{\theta}_n}$ of $\tau(\theta) = \frac{1}{\theta}$ is thus

$$\frac{\frac{1}{\hat{\theta}_n} - \frac{1}{\theta}}{\sqrt{CRLB}} = \frac{\frac{1}{\hat{\theta}_n} - \frac{1}{\theta}}{\frac{1}{\sqrt{n}\theta}} \xrightarrow{d} Z \sim \mathrm{N}(0,1).$$

(f) This exercise is similar to Exercise 25(b). Given that $T = \frac{S}{n} = \frac{1}{n}\sum_{i=1}^n \ln(1+X_i)$ was an UMVUE for $1/\theta$, it seems reasonable to figure out whether n/S can be an UMVUE for θ. It is clearly a function of the complete sufficient statistic so it only remains to check whether it is unbiased. Because the pdf of S is given by

$$f_S(s) = \frac{1}{\left(\frac{1}{\theta}\right)^n \Gamma(n)} s^{n-1} e^{-\frac{s}{\frac{1}{\theta}}} = \frac{1}{\left(\frac{1}{\theta}\right)^n \Gamma(n)} s^{n-1} e^{-\theta s} \qquad s > 0,$$

we find

$$\begin{aligned}
\mathbb{E}\left(\frac{n}{S}\right) &= \int_0^\infty \frac{n}{s}\frac{1}{\left(\frac{1}{\theta}\right)^n \Gamma(n)} s^{n-1} e^{-\theta s} ds = n \int_0^\infty \left(\frac{\Gamma(n-1)}{\Gamma(n-1)}\right) \frac{1}{\left(\frac{1}{\theta}\right)^n \Gamma(n)} s^{(n-1)-1} e^{-\theta s} ds \\
&= n\theta \frac{\Gamma(n-1)}{\Gamma(n)} \int_0^\infty \underbrace{\frac{1}{\left(\frac{1}{\theta}\right)^{n-1} \Gamma(n-1)} s^{(n-1)-1} e^{-\theta s}}_{\text{pdf of GAM}(\frac{1}{\theta}, n-1)} ds \\
&= n\theta \frac{\Gamma(n-1)}{\Gamma(n)} = \frac{n}{n-1}\theta.
\end{aligned}$$

An UMVUE for θ is thus $\frac{n-1}{S} = \frac{n-1}{\sum_{i=1}^n \ln(1+X_i)}$.