Solutions to Selected Exercises from Chapter 12
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Exercise 1

(a) If $X_1, \dots, X_n \overset{\text{i.i.d.}}{\sim} \text{N}(\mu, 1)$, then $\frac{\bar{X}-\mu}{1/\sqrt{n}} = \sqrt{n}(\bar{X}-\mu) \sim \text{N}(0,1)$. For the rejection region A, we realize that

$$\alpha = \mathbb{P}\left(\frac{\bar{X}-\mu}{1/\sqrt{n}} < -z_{1-\alpha} \,\middle|\, \mu = 20 \right) = \mathbb{P}\left(\bar{X} < \mu - \frac{z_{1-\alpha}}{\sqrt{n}} \,\middle|\, \mu = 20 \right).$$

Using $z_{0.95} \approx 1.645$ and filling in the values, we find the reject region $A = \{\bar{x} \,|\, -\infty < \bar{x} \leq 19.589\}$. For rejection region B we will reject in the right tail of the distribution. The calculation

$$\alpha = \mathbb{P}\left(\frac{\bar{X}-\mu}{1/\sqrt{n}} > z_{1-\alpha} \,\middle|\, \mu = 20 \right) = \mathbb{P}\left(\bar{X} > \mu + \frac{z_{1-\alpha}}{\sqrt{n}} \,\middle|\, \mu = 20 \right),$$

shows that the rejection region B takes the form $\{\bar{x} \,|\, 20.411 \leq \bar{x} < \infty\}$.

(b) We need the probability to not reject even though the null hypothesis is false. For the critical region A, we have

$$\begin{aligned}\mathbb{P}(\text{TII}) &= \mathbb{P}(\bar{X} > 19.589 | \mu = 21) = \mathbb{P}\left(\frac{\bar{X}-21}{1/\sqrt{16}} > \frac{19.589-21}{1/\sqrt{16}} \,\middle|\, \mu = 21 \right) \\ &= \mathbb{P}(Z > -5.64) = \Phi(5.64) \approx 1.\end{aligned}$$

For critical region B, the probability of a Type II error is

$$\begin{aligned}\mathbb{P}(\text{TII}) &= \mathbb{P}(\bar{X} < 20.411 | \mu = 21) = \mathbb{P}\left(\frac{\bar{X}-21}{1/\sqrt{16}} < \frac{20.411-21}{1/\sqrt{16}} \,\middle|\, \mu = 21 \right) \\ &= \mathbb{P}(Z < -2.36) \approx 0.01.\end{aligned}$$

Comparing the probabilities of these Type II errors, we conclude that critical region A is unreasonable for this alternative.

(c) For critical region A, we have

$$\begin{aligned}\mathbb{P}(\text{TII}) &= \mathbb{P}(\bar{X} > 19.589 | \mu = 19) = \mathbb{P}\left(\frac{\bar{X}-19}{1/\sqrt{16}} > \frac{19.589-19}{1/\sqrt{16}} \,\middle|\, \mu = 21 \right) \\ &= \mathbb{P}(Z > 2.36) = \Phi(-2.36) \approx 0.01,\end{aligned}$$

whereas for critical region B we get

$$\begin{aligned}\mathbb{P}(\text{TII}) &= \mathbb{P}(\bar{X} < 20.411|\mu = 19) = \mathbb{P}\left(\frac{\bar{X} - 19}{1/\sqrt{16}} < \frac{20.411 - 19}{1/\sqrt{16}}\middle| \mu = 21\right) \\ &= \mathbb{P}(Z < 5.64) \approx 1.\end{aligned}$$

This time the unreasonable critical region is region B.

(d) We have

$$\mathbb{P}\left(\bar{X} \in (A \cup B)\middle|\mu = 20\right) = \mathbb{P}\left(\bar{X} \in A\middle|\mu = 20\right) + \mathbb{P}\left(\bar{X} \in B\middle|\mu = 20\right) = 0.05 + 0.05 = 0.1,$$

since the critical regions A and B are disjoint (probabilities add up). The significance level for the test with rejection region $A \cup B$ is thus 10%.

(e) The condition $|\mu - 20| = 1$ implies either $\mu = 19$ or $\mu = 21$. We first consider $\mu = 19$. Since A and B are disjoint, the probability to reject the null equals

$$\begin{aligned}\mathbb{P}('reject'|\mu = 19) &= \mathbb{P}(\bar{X} \in A|\mu = 19) + \mathbb{P}(\bar{X} \in B|\mu = 19) \\ &= \mathbb{P}(\bar{X} \leq 19.589|\mu = 19) + \mathbb{P}(\bar{X} \geq 20.411|\mu = 19) \\ &= \mathbb{P}\left(Z \leq \frac{19.589 - 19}{1/\sqrt{16}}\right) + \mathbb{P}\left(Z \geq \frac{20.411 - 19}{1/\sqrt{16}}\right) = \Phi(2.356) + \Phi(-5.644) \\ &\approx 0.9908.\end{aligned}$$

The probability for a Type II error is thus $1 - 0.9908 \approx 0.92\%$. We can perform a similar calculation for $\mu = 21$, that is

$$\begin{aligned}\mathbb{P}('reject'|\mu = 21) &= \mathbb{P}(\bar{X} \in A|\mu = 21) + \mathbb{P}(\bar{X} \in B|\mu = 21) \\ &= \mathbb{P}(\bar{X} \leq 19.589|\mu = 21) + \mathbb{P}(\bar{X} \geq 20.411|\mu = 21) \\ &= \mathbb{P}\left(Z \leq \frac{19.589 - 21}{1/\sqrt{16}}\right) + \mathbb{P}\left(Z \geq \frac{20.411 - 21}{1/\sqrt{16}}\right) = \Phi(-5.644) + \Phi(2.356) \\ &\approx 0.9908.\end{aligned}$$

The probability for a Type II error is thus $1 - 0.9908 \approx 0.92\%$. We see that rejection region $A \cup B$ controls the Type II error for alternatives that are both lower and higher than the value under the null.

Exercise 3

(a) The value of the Z-statistic is equal to $z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{11-12}{2/\sqrt{20}} \approx -2.236$. According to the alternative hypothesis, we will reject in the left tail of the distribution. The critical value is $-z_{0.99} \approx -2.326$. Since $z_0 > -2.236$, we do not reject H_0.

(b) Making use of the power function $\pi(\mu)$ as defined in Theorem 12.3.1, we find that the probability of a Type II error is

$$\begin{aligned}\beta &= 1 - \pi(10.5) = 1 - \Phi\left(-z_{1-\alpha} + \frac{\mu_0 - 10.5}{\sigma/\sqrt{n}}\right) = 1 - \Phi\left(-2.326 + \frac{12 - 10.5}{2/\sqrt{20}}\right) \\ &= 1 - \Phi(1.028) \approx 0.15.\end{aligned}$$

(c) We use point 4. of Theorem 12.3.1. With $z_{1-\alpha} = z_{0.99} = 2.326$ and $z_{1-\beta} = z_{0.9} = 1.282$, the required sample size is

$$n \geq \frac{(z_{1-\alpha} + z_{1-\beta})^2 \sigma^2}{(\mu_0 - \mu)^2} = \frac{(2.326 + 1.282)^2 4}{(12 - 10.5)^2} = 23.143.$$

At least $n = 24$ observations are required.

(d) The numerical value of the t-test is equal to $t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{11-12}{4/\sqrt{20}} = -1.118$. We should reject the null hypothesis whenever $t_0 < -t_{0.99}$, where $t_{0.99}$ denotes the 99% quantile of t-distribution with 19 degrees of freedom. We find $t_{0.99} \approx 2.539$. Since $t_0 > -2.539$, we do not reject the null hypothesis.

(e) According to Theorem 12.3.3, we can use the test statistic $v_0 = \frac{(n-1)s^2}{\sigma^2} = \frac{(20-1)\times 16}{9} \approx 33.78$. For the given alternative, we should reject whenever $v_0 > \chi^2_{0.99}$, where $\chi^2_{0.99}$ denotes the 99% quantile of the χ^2-distribution with 19 degrees of freedom. We have $\chi^2_{0.99} \approx 36.19$ and hence do not reject the null hypothesis.

(f) According to Theorem 12.3.3, the power function is $\pi(\sigma^2) = 1 - H\big(\frac{\sigma_0^2}{\sigma^2}\chi^2_{1-\alpha}(n-1); n-1\big)$, where $H(x; n-1)$ denotes the CDF of the $\chi^2(n-1)$ distribution. We write

$$\begin{aligned}
1 - H\left(\frac{\sigma_0^2}{\sigma^2}\chi^2_{1-\alpha}(n-1); n-1\right) &\geq 0.9 \\
H\left(\frac{\sigma_0^2}{\sigma^2}\chi^2_{1-\alpha}(n-1); n-1\right) &\leq 0.1 \\
\frac{\sigma_0^2}{\sigma^2}\chi^2_{1-\alpha}(n-1) &\leq \chi^2_{0.1}(n-1) \\
\frac{\chi^2_{0.1}(n-1)}{\chi^2_{1-\alpha}(n-1)} &\geq \frac{\sigma_0^2}{\sigma^2} \\
\frac{\chi^2_{0.1}(n-1)}{\chi^2_{0.99}(n-1)} &\geq \frac{9}{18} = \frac{1}{2}.
\end{aligned}$$

Going through Table 4, it can be seen that the above holds if $n - 1 \geq 60$. Hence at least $n = 61$ observations are required (note that Table 4 does not contain values for degrees of freedom between 50 and 60, though). The probability of a Type II error if $\sigma^2 = 18$ is

$$\begin{aligned}
\beta = 1 - \pi(\sigma^2) &= H\left(\frac{\sigma_0^2}{\sigma^2}\chi^2_{1-\alpha}(n-1); n-1\right) = H\left(\frac{9}{18}\chi^2_{1-\alpha}(n-1); n-1\right) \\
&= H\left(\frac{1}{2}\chi^2_{0.99}(60); 60\right) = H(44.19; 60)
\end{aligned}$$

whose value is not in Table 5, but could be computed with the approximation given there for large degrees of freedom.

Exercise 4

The pdf of X is $f(x; p) = \mathbb{P}(X = x) = p(1-p)^{x-1}$ for $x = 1, 2, \ldots$, since there are $x - 1$ unsuccessful tosses with probability $(1-p)^{x-1}$ before the first successful toss with probability p.

(a) For the probability of a Type I error we need the probability to reject when H_0 is true. We thus use $p = 0.80$, or

$$\begin{aligned}\mathbb{P}(X \geq 3 \,|\, p = 0.80) &= 1 - \mathbb{P}(X = 1|p = 0.80) - \mathbb{P}(X = 2|p = 0.80) \\ &= 1 - p(1-p)^0 - p(1-p) = 1 - p - p(1-p) = (1-p)^2 = 0.20^2 = 0.04.\end{aligned}$$

(b) We need the probability to *not* reject when $p = 0.20$ and $p = 0.30$. For general p, the probability of a Type II error is

$$\begin{aligned}\mathbb{P}(X < 3|p) &= \mathbb{P}(X = 1|p) + \mathbb{P}(X = 2|p) = p(1-p)^0 + p(1-p) = p + p(1-p) \\ &= p(2-p).\end{aligned}$$

Denoting the probability of a Type II error by β, we have $\beta = 0.20(2 - 0.20) = 0.36$ and $\beta = 0.30(2 - 0.30) = 0.51$, for $p = 0.20$ and $p = 0.30$ respectively.

(c) Let us calculate the rejection probability for arbitrary p. We have

$$\begin{aligned}\mathbb{P}\left(X \in \{1, 14, 15, \ldots\}|p\right) &= \mathbb{P}(X = 1|p) + \sum_{x=14}^{\infty} \mathbb{P}(X = x|p) \\ &= p(1-p)^0 + \sum_{x=14}^{\infty} p(1-p)^{x-1} = p + (1-p)^{13} \sum_{x=0}^{\infty} p(1-p)^x \\ &= p + (1-p)^{13} \frac{p}{1-(1-p)} = p + (1-p)^{13},\end{aligned}$$

using the following result on geometric series: $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$ for $|r| < 1$. We can find the probability of a type I error by evaluating the expression above for $p = 0.30$, that is $0.30 + 0.70^{13} = 0.310$. For the type II error we need the probability to not reject. So denoting the probability of the type II error by β, we find

$$\beta = \mathbb{P}(X \notin \{1, 14, 14, \ldots\}|p) = 1 - \left(p + (1-p)^{13}\right)$$

whenever $p \neq 0.30$. For $p = 0.20$, this gives $\beta = 1 - (0.20 + 0.80^{13}) = 0.745$. For $p = 0.80$, we obtain $\beta = 1 - (0.80 + 0.20^{13}) = 0.200$.

Exercise 9

(a) We first compute the pooled variance estimate

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} = \frac{8 \cdot 36 + 8 \cdot 45}{16} = 40.5.$$

The t-statistic now takes the value $t = \frac{\bar{y} - \bar{x}}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{10-16}{\sqrt{40.5\left(\frac{1}{9} + \frac{1}{9}\right)}} = -2$. Under the null hypothesis, this statistic follows a t-distribution with $n_1 + n_2 - 2 = 9 + 9 - 2 = 16$ degrees of freedom. If $t_{0.95} \approx 1.756$ denotes the 95% quantile of this distribution, then we will reject if $|t| > 1.756$. We have $-2 < -1.756$ and therefore reject the null.

(b) From Equation (11.5.14) we estimate the degrees of freedom as

$$\nu = \frac{\left(s_1^2/n_1 + s_2^2/n_2\right)^2}{\frac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \frac{\left(s_2^2/n_2\right)^2}{n_2 - 1}} = \frac{(36/9 + 45/9)^2}{\frac{(36/9)^2}{8} + \frac{(45/9)^2}{8}} = 15.805$$

and compute the corresponding critical value by linear interpolation

$$t_{0.95} = t_{0.95}(15) + 0.805\big(t_{0.95}(16) - t_{0.95}(15)\big) = 1.753 + 0.805(1.746 - 1.753) = 1.747.$$

We will thus reject the null hypothesis if the absolute value of the observed test statistic exceeds 1.747. A calculation of this test statistic gives

$$t_0 = \frac{\bar{y} - \bar{x}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_1^2}{n_2}}} = \frac{10 - 16}{\sqrt{\frac{36}{9} + \frac{45}{9}}} = -2,$$

and we therefore reject the null hypothesis.

(c) The value of the test statistic is $t_0 = \frac{\bar{y}-\bar{x}}{s_D/\sqrt{n}} = \frac{10-16}{9/\sqrt{9}} = -2$. We should compare this outcome with the 95% quantile of the t-distribution with $(9-1) = 8$ degrees of freedom. The implied critical value is 1.860. Since $|-2| > 1.860$ we reject the null hypothesis.

(d) We use Theorem 12.3.4. We compute the test statistic as $f_0 = \frac{s_1^2}{s_2^2} = \frac{36}{45} = 0.8$. If we let $f_{1-\alpha}(n_2-1, n_1-1)$ denote the $(1-\alpha)$-quantile of the F-distribution with (n_2-1) and (n_1-1) degrees of freedom, then we should reject whenever $f_0 \leq \frac{1}{f_{1-\alpha}}$. We find $\frac{1}{f_{0.95}} = \frac{1}{3.44} = 0.29$ and do not reject H_0.

(e) We have to derive the power function at $\frac{\sigma_2^2}{\sigma_1^2} = 1.33$. For general $\frac{\sigma_2^2}{\sigma_1^2}$, we find

$$\begin{aligned}\pi\left(\frac{\sigma_2^2}{\sigma_1^2}\right) &= \mathbb{P}\left(\frac{S_1^2}{S_2^2} \leq \frac{1}{f_{1-\alpha}} \,\middle|\, \frac{\sigma_2^2}{\sigma_1^2}\right) = \mathbb{P}\left(\frac{S_1^2}{S_2^2}\frac{\sigma_2^2}{\sigma_1^2} \leq \frac{1}{f_{1-\alpha}}\frac{\sigma_2^2}{\sigma_1^2} \,\middle|\, \frac{\sigma_2^2}{\sigma_1^2}\right) \\ &= \mathbb{P}\left(\frac{[(n_1-1)S_1^2/\sigma_1^2]/(n_1-1)}{[(n_2-1)S_2^2/\sigma_2^2]/(n_2-1)} \leq \frac{1}{f_{1-\alpha}}\frac{\sigma_2^2}{\sigma_1^2} \,\middle|\, \frac{\sigma_2^2}{\sigma_1^2}\right) = \mathbb{P}\left(F(n_1-1, n_2-1) \leq \frac{1}{f_{1-\alpha}}\frac{\sigma_2^2}{\sigma_1^2}\right),\end{aligned}$$

where $F(n_1-1, n_2-1)$ denotes an F-distributed random variable with (n_1-1, n_2-1) degrees of freedom. After calculating $\frac{1}{f_{1-\alpha}}\frac{\sigma_2^2}{\sigma_1^2} = 0.387$ we find this probability to be equal approximately 0.1.

Exercise 11

(a) We use the Neyman-Pearson Lemma, Theorem 12.6.1 from B&E. We should reject the nulll hypothesis when $\lambda(x;1,2) = \frac{f(x;1)}{f(x;2)} = \frac{1}{2x}$ is small, or equivalently for large x. To find the most powerful test with significance level α, we require that

$$\mathbb{P}(X \geq c|\theta = 1) = \int_c^1 f(x;1)dx = 1 - c = \alpha.$$

The most powerful critical region of size α for testing $H_0 : \theta = 1$ versus $H_a : \theta = 2$ is thus $C^* = \{x \,|\, x \geq 1 - \alpha\}$. For the given significance level we would reject when $x > 0.95$.

(b) The power function is

$$\pi(\theta) = \mathbb{P}(X \geq 0.95|\theta) = \int_{0.95}^1 f(x;\theta)dx = x^\theta\big|_{0.95}^1 = 1 - (0.95)^\theta.$$

For $\theta = 2$ we have $\pi(2) = 1 - 0.95^2 = 0.0975$.

(c) The joint pdf of $X_1, \ldots, X_n = \prod_{i=1}^n \theta x_i^{\theta-1} = \theta^2 \left(\prod_{i=1}^n x_i\right)^{\theta-1}$ and hence

$$\lambda(x_1, \ldots, x_n; 1, 2) = \frac{1}{2^n \prod_{i=1}^n x_i}.$$

We should reject the null hypothesis for small values of $\lambda(x_1, \ldots, x_n; 1, 2)$. This coincides with large values of $\prod_{i=1}^n x_i$. The distribution of $\prod_{i=1}^n X_i$ is difficult to establish. However, we can apply additional monotone transformations. Note that rejection for large $\prod_{i=1}^n x_i$ is equivalent to rejection for large $\sum_{i=1}^n \ln(x_i)$, is equivalent to rejection for small $\sum_{i=1}^n - \ln(x_i)$. This will turn out to be helpful because if X has pdf $f(x; \theta)$, then $Y = -\ln(X)$ had pdf

$$f_Y(y) = f_X(e^{-y}) \left| -e^{-y} \right| = \theta \left(e^{-y}\right)^{\theta-1} e^{-y} = \theta e^{-\theta y}, \qquad y > 0.$$

Apparently, Y is EXP$(1/\theta)$ distributed and thus $-2\theta \sum_{i=1}^n \ln(X_i) = \frac{2n\bar{Y}}{1/\theta} \sim \chi^2(2n)$. Since we agreed to reject for small values of $\sum_{i=1}^n - \ln(x_i)$, we compute the critical value from

$$\mathbb{P}\left(-\sum_{i=1}^n \ln X_i \leq c \,\middle|\, \theta = 1 \right) = \mathbb{P}\left(-2\sum_{i=1}^n \ln X_i \leq 2c \right) = \alpha.$$

We find $c = \chi^2_\alpha/2$, where χ^2_α denotes the $100\alpha\%$ quantile of the $\chi^2(2n)$ distribution. The most powerful critical region of size α for testing $H_0 : \theta = 1$ versus $H_a : \theta = 2$ is thus $C^* = \left\{ (x_1, \ldots, x_n) \,\middle|\, -\sum_{i=1}^n \ln x_i \leq \frac{\chi^2_\alpha}{2} \right\}$.

Exercise 12

(a) We use the Neyman-Pearson Lemma, Theorem 12.6.1 from B&E. Using the pdf $f(x; \mu) = \frac{e^{-\mu}\mu^x}{x!}$ we find

$$\lambda(x; \mu_0, \mu_1) = \frac{\frac{e^{-\mu_0}\mu_0^x}{x!}}{\frac{e^{-\mu_1}\mu_1^x}{x!}} = e^{\mu_1 - \mu_0} \left(\frac{\mu_0}{\mu_1}\right)^x.$$

We should reject when $\lambda(x; \mu_0, \mu_1)$ is small, or equivalently when

$$e^{\mu_1 - \mu_0} \left(\frac{\mu_0}{\mu_1}\right)^x \leq k_1 \quad \Rightarrow \quad \left(\frac{\mu_0}{\mu_1}\right)^x \leq \frac{k_1}{e^{\mu_1 - \mu_0}} = k_2 \quad \Rightarrow \quad x \ln\left(\frac{\mu_0}{\mu_1}\right) \leq \ln(k_2) = k_3$$
$$\Rightarrow x \geq \frac{k_3}{\ln\left(\frac{\mu_0}{\mu_1}\right)} = c,$$

where k_1, k_2, k_3 and c are the constants to be determined to control size. Also note that $\ln(\mu_0/\mu_1) < 0$ because $\mu_1 > \mu_0$ is given in the exercise. To obtain the correct significance level we should define the rejection region such that $\mathbb{P}(X > c | \mu = \mu_0) = 1 - F(c; \mu_0) = \alpha$. The critical value c is thus $F^{-1}(1 - \alpha; \mu_0)$. The most powerful critical region of size α for testing $H_0 : \mu = \mu_0$ versus $H_a : \mu = \mu_1$ is thus $C^* = \left\{ x \,\middle|\, x \geq F^{-1}(1 - \alpha; \mu_0) \right\}$.

(b) The joint pdf of $X_1, \ldots, X_n$ is $f(x_1, \ldots, x_n; \mu) = \prod_{i=1}^n \frac{e^{-\mu}\mu^{x_i}}{x_i!} = \frac{e^{-n\mu}\mu^{\sum_{i=1}^n x_i}}{(\prod_{i=1}^n x_i!)}$. We find

$$\lambda(x; \mu_0, \mu_1) = \frac{\frac{e^{-n\mu_0}\mu_0^{\sum_{i=1}^n x_i}}{(\prod_{i=1}^n x_i!)}}{\frac{e^{-n\mu_1}\mu_1^{\sum_{i=1}^n x_i}}{(\prod_{i=1}^n x_i!)}} = e^{n(\mu_1 - \mu_0)} \left(\frac{\mu_0}{\mu_1}\right)^{\sum_{i=1}^n x_i}.$$

We should reject when $\lambda(x;\mu_0,\mu_1)$ is small, or equivalently when

$$e^{n(\mu_1-\mu_0)}\left(\frac{\mu_0}{\mu_1}\right)^{\sum_{i=1}^n x_i} \leq k_1 \quad\Rightarrow\quad \left(\frac{\mu_0}{\mu_1}\right)^{\sum_{i=1}^n x_i} \leq \frac{k_1}{e^{n(\mu_1-\mu_0)}} = k_2$$
$$\Rightarrow\quad \left(\sum_{i=1}^n x_i\right)\ln\left(\frac{\mu_0}{\mu_1}\right) \leq \ln(k_2) = k_3 \quad\Rightarrow\quad \left(\sum_{i=1}^n x_i\right) \geq \frac{k_3}{\ln\left(\frac{\mu_0}{\mu_1}\right)} = c,$$

where k_1, k_2, k_3 and c are the constants to be determined to control size. If $X_1,\ldots,X_n \sim \mathrm{POI}(\mu)$, then $\sum_{i=1}^n X_i \sim \mathrm{POI}(n\mu)$ (see Example 6.4.5). To obtain the correct significance level we should define the rejection region such that $\mathbb{P}(\sum_{i=1}^n X_i > c \,|\, \mu = \mu_0) = 1 - F(c; n\mu_0) = \alpha$. The critical value c is thus $F^{-1}(1-\alpha; n\mu_0)$. The most powerful critical region of size α for testing $H_0 : \mu = \mu_0$ versus $H_a : \mu = \mu_1$ is thus $C^* = \left\{(x_1,\ldots,x_n) \,\middle|\, \sum_{i=1}^n x_i \geq F^{-1}(1-\alpha; n\mu_0)\right\}$.

Exercise 16

Suppose we would test $H_0 : \theta = \theta_0$ versus $H_a : \theta = \theta_1$ with $\theta_1 > \theta_0$. Having simple hypothesis we could now use the Neyman-Pearson Lemma, Theorem 12.6.1 from B&E. We would get the joint pdf

$$f(x_1,\ldots,x_n|\theta) = \prod_{i=1}^n \frac{3x_i^2}{\theta}e^{-x_i^3/\theta} = \frac{3^n}{\theta^n}\left(\prod_{i=1}^n x_i\right)^2 e^{-\frac{1}{\theta}\sum_{i=1}^n x_i^3}$$

and

$$\lambda(x_1,\ldots,x_n;\theta_0,\theta_1) = \frac{\frac{3^n}{\theta_0^n}\left(\prod_{i=1}^n x_i\right)^2 e^{-\frac{1}{\theta_0}\sum_{i=1}^n x_i^3}}{\frac{3^n}{\theta_1^n}\left(\prod_{i=1}^n x_i\right)^2 e^{-\frac{1}{\theta_1}\sum_{i=1}^n x_i^3}} = \left(\frac{\theta_1}{\theta_0}\right)^n e^{\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)\sum_{i=1}^n x_i^3}.$$

The null hypothesis should be rejected if

$$\left(\frac{\theta_1}{\theta_0}\right)^n e^{\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)\sum_{i=1}^n x_i^3} \leq k_1 \quad\Rightarrow\quad e^{\left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)\sum_{i=1}^n x_i^3} \leq k_1\left(\frac{\theta_0}{\theta_1}\right)^n = k_2$$
$$\Rightarrow\quad \left(\frac{1}{\theta_1}-\frac{1}{\theta_0}\right)\sum_{i=1}^n x_i^3 \leq \ln k_2 = k_3 \quad\Rightarrow\quad \sum_{i=1}^n x_i^3 \geq \frac{k_3}{\frac{1}{\theta_1}-\frac{1}{\theta_0}} = c,$$

where k_1, k_2, k_3 and c are the constants to be determined to control the Type I error. We have to find the distribution of $\sum_{i=1}^n X_i^3$. Let X have pdf $f(x;\theta)$, then $Y = X^3$ has the pdf

$$f_Y(y) = f_X\left(y^{\frac{1}{3}}\right)\left|\frac{1}{3}y^{-\frac{2}{3}}\right| = \frac{3y^{\frac{2}{3}}}{\theta}e^{-\frac{y}{\theta}}\frac{1}{3}y^{-\frac{2}{3}} = \frac{1}{\theta}e^{-\frac{y}{\theta}}, \qquad y > 0.$$

We conclude that $Y \sim \mathrm{EXP}(\theta)$ and realize that $\frac{2}{\theta}\sum_{i=1}^n X_i^3 = \frac{2n\bar{Y}}{\theta} \sim \chi^2(2n)$. Size control requires

$$\mathbb{P}\left(\sum_{i=1}^n X_i^3 \geq c \,\middle|\, \theta = \theta_0\right) = \mathbb{P}\left(\frac{2}{\theta_0}\sum_{i=1}^n X_i^3 \geq \frac{2c}{\theta_0}\right) = \alpha,$$

or $\frac{2c}{\theta_0} = \chi^2_{1-\alpha}$, hence $c = \frac{\theta_0\chi^2_{1-\alpha}}{2}$, where $\chi^2_{1-\alpha}$ denotes the $100(1-\alpha)\%$ quantile of the $\chi^2(2n)$ distribution. The most powerful critical region of size α for testing $H_0 : \theta = \theta_0$ versus $H_a : \theta = \theta_1$ (where $\theta_1 > \theta_0$) is thus $C^* = \left\{(x_1,\ldots,x_n) \,\middle|\, \sum_{i=1}^n x_i^3 \geq \frac{\theta_0\chi^2_{1-\alpha}}{2}\right\}$. Since the critical region C^*

does not depend on a specific value of $\theta_1 > \theta_0$, it corresponds to a uniformly most powerful test for $H_0 : \theta = \theta_0$ against $H_a : \theta > \theta_0$.

Exercise 17

(a) We first consider $H_0 : \sigma = \sigma_0$ and $H_a : \sigma = \sigma_1$ with $\sigma_1 > \sigma_0$. We use the Neyman-Pearson Lemma, Theorem 12.6.1 from B&E. The joint pdf

$$f(x_1, \ldots, x_n; \sigma) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x_i^2}{2\sigma^2}} = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2},$$

and

$$\lambda(x_1, \ldots, x_n; \sigma_0, \sigma_1) = \frac{(2\pi\sigma_0^2)^{-n/2} e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^n x_i^2}}{(2\pi\sigma_1^2)^{-n/2} e^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^n x_i^2}} = \left(\frac{\sigma_1}{\sigma_0}\right)^n e^{\left(\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2}\right)\sum_{i=1}^n x_i^2}.$$

H_0 is rejected if $\lambda(x_1, \ldots, x_n; \sigma_0, \sigma_1)$ is too small, or equivalently,

$$\left(\frac{\sigma_1}{\sigma_0}\right)^n e^{\left(\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2}\right)\sum_{i=1}^n x_i^2} \le k_1 \quad \Rightarrow \quad e^{\left(\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2}\right)\sum_{i=1}^n x_i^2} \le k_1 \left(\frac{\theta_0}{\theta_1}\right)^n = k_2$$

$$\Rightarrow \quad \left(\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2}\right)\sum_{i=1}^n x_i^2 \le \ln k_2 = k_3 \quad \Rightarrow \quad \sum_{i=1}^n x_i^2 \ge \frac{k_3}{\frac{1}{2\sigma_1^2} - \frac{1}{2\sigma_0^2}} = c$$

where k_1, k_2, k_3 and c are the constants to be determined to control the Type I error. If $X_1, \ldots, X_n \sim \mathrm{N}(0, \sigma^2)$, then $\frac{\sum_{i=1}^n X_i^2}{\sigma^2} \sim \chi^2(n)$. We can thus control the probability of a Type I error by requiring

$$\mathbb{P}\left(\sum_{i=1}^n X_i^2 \ge c \,\middle|\, \sigma = \sigma_0\right) = \mathbb{P}\left(\frac{\sum_{i=1}^n X_i^2}{\sigma_0^2} \ge \frac{c}{\sigma_0^2}\right) = \alpha.$$

This implies $\frac{c}{\sigma_0^2} = \chi^2_{1-\alpha}$ or $c = \sigma_0^2 \chi^2_{1-\alpha}$, where $\chi^2_{1-\alpha}$ denotes the $100(1-\alpha)\%$ quantile of the $\chi^2(n)$ distribution. The most powerful critical region of size α for testing $H_0 : \sigma = \sigma_0$ versus $H_a : \sigma = \sigma_1$ (where $\sigma_1 > \sigma_0$) is thus $C^* = \left\{(x_1, \ldots, x_n) \,\middle|\, \sum_{i=1}^n x_i^2 \ge \sigma_0^2 \chi^2_{1-\alpha}\right\}$. Since the critical region C^* does not depend on a specific value of $\sigma_1 > \sigma_0$, it corresponds to a uniformly most powerful test for $H_0 : \sigma = \sigma_0$ against $H_a : \sigma > \sigma_0$.

(b) The power function is

$$\pi(\sigma) = \mathbb{P}\left(\sum_{i=1}^n X_i^2 \ge \sigma_0^2 \chi^2_{1-\alpha} \,\middle|\, \sigma\right) = \mathbb{P}\left(\frac{\sum_{i=1}^n X_i^2}{\sigma^2} \ge \frac{\sigma_0^2}{\sigma^2}\chi^2_{1-\alpha} \,\middle|\, \sigma\right) = 1 - H\left(\frac{\sigma_0^2}{\sigma^2}\chi^2_{1-\alpha}; n\right),$$

where $H(x; n)$ denotes the CDF of the $\chi^2(n)$ distribution.

(c) $\pi(4) = 1 - H\left(\frac{1}{4}\chi^2_{0.995}; 20\right) = 1 - H(10.00; 20) = 1 - 0.032 = 0.968.$

Exercise 27

(a) The joint distribution of $X_1, \ldots, X_n$ is $f(\boldsymbol{x};\theta) = \prod_{i=1}^n \frac{1}{\theta} e^{-x_i/\theta} = \theta^{-n} e^{-n\bar{x}/\theta}$. If the null is true, then θ has to equal θ_0. The unrestricted ML estimate is $\hat{\theta} = \bar{x}$ (see Example 9.2.7). This implies

$$\lambda(\boldsymbol{x}) = \frac{\max_{\theta\in\Omega_0} f(\boldsymbol{x};\theta)}{\max_{\theta\in\Omega} f(\boldsymbol{x};\theta)} = \frac{f(\boldsymbol{x};\theta_0)}{f(\boldsymbol{x};\hat{\theta})} = \frac{\theta_0^{-n} e^{-n\bar{x}/\theta_0}}{\bar{x}^{-n} e^{-n}} = \left(\frac{\bar{x}}{\theta_0}\right)^n e^{n(1-\frac{\bar{x}}{\theta_0})},$$

and

$$-2\ln\big(\lambda(\boldsymbol{x})\big) = -2n\left(1 - \frac{\bar{x}}{\theta_0} + \ln\left(\frac{\bar{x}}{\theta_0}\right)\right).$$

The null hypothesis imposes 1 restriction on the parameter space. According to Equation (12.8.3), an approximate size α test is to reject H_0 if

$$-2n\left(1 - \frac{\bar{x}}{\theta_0} + \ln\left(\frac{\bar{x}}{\theta_0}\right)\right) \geq \chi^2_{1-\alpha}(1).$$

(b) The parameter space is $\Omega = [\theta_0, \infty)$. There is still only the single parameter value θ_0 possible under the null. We now compute the ML estimate for $\theta \in [\theta_0, \infty)$. From part (a) we have the likelihood $L(\theta) = \theta^{-n} e^{-n\bar{x}/\theta}$ which implies the log-likelihood $\ln L(\theta) = -n\ln(\theta) - \frac{n\bar{x}}{\theta}$. The first derivative is

$$\frac{d}{d\theta}\ln L(\theta) = -\frac{n}{\theta} + \frac{n\bar{x}}{\theta^2} = -\frac{n}{\theta^2}(\theta - \bar{x}) = \begin{cases} - & \text{if } \theta > \bar{x} \\ + & \text{if } \theta < \bar{x}. \end{cases}$$

For this we conclude that the maximum will equal $\bar{x}$ when $\theta_0 < \bar{x}$, or θ_0 when $\theta_0 > \bar{x}$. We conclude that

$$\lambda(\boldsymbol{x}) = \frac{\max_{\theta\in\Omega_0} f(\boldsymbol{x};\theta)}{\max_{\theta\in\Omega} f(\boldsymbol{x};\theta)} = \begin{cases} \left(\frac{\bar{x}}{\theta_0}\right)^n e^{n(1-\frac{\bar{x}}{\theta_0})} & \text{if } \bar{x}/\theta_0 > 1, \\ 1 & \text{if } \bar{x}/\theta_0 < 1. \end{cases}$$

Now recall that we should reject the null hypothesis for small values of $\lambda(\boldsymbol{x})$ where 'small' should be quantified based on the maximum probability of a Type I error. Under the null, we have

$$\begin{aligned}\mathbb{P}(\lambda(\boldsymbol{X}) < 1|\theta_0) &= \mathbb{P}(\bar{X}/\theta_0 > 1|\theta_0) = \mathbb{P}\left(\frac{2n\bar{X}}{\theta_0} > 2n \,\middle|\, \theta_0\right) = \mathbb{P}\left(\chi^2(2n) > 2n\right) \\ &= 1 - \mathbb{P}\left(\chi^2(2n) \leq 2n\right).\end{aligned}$$

From Table 5 in the Appendix C of B&E we can see that this probability varies around 50%. For typical sizes (say 1%, 5%, 10%) we will thus find ourselves in the case where $\bar{x}/\theta_0 > 1$. We will thus assume that $\alpha < \mathbb{P}(\lambda(\boldsymbol{X}) < 1|\theta_0)$ (and thus $\bar{x}/\theta_0 > 1$).

The rejection regions are of the following forms

$$\left(\frac{\bar{x}}{\theta_0}\right)^n e^{n(1-\frac{\bar{x}}{\theta_0})} \leq k \quad\Rightarrow\quad \left(\frac{\bar{x}}{\theta_0}\right) e^{(1-\frac{\bar{x}}{\theta_0})} \leq k^{1/n} = k_1 \quad\Rightarrow\quad \left(\frac{\bar{x}}{\theta_0}\right) e^{-\frac{\bar{x}}{\theta_0}} \leq k_1 e^{-1} = k_2,$$

where k, k_1 and k_2 are the constants to be determined to control the Type I error. To analysis the inequality $\left(\frac{\bar{x}}{\theta_0}\right) e^{-\frac{\bar{x}}{\theta_0}} \leq k_2$ in more detail, we define the function $f(y) = ye^{-y}$ such that $f\left(\frac{\bar{x}}{\theta_0}\right) = \left(\frac{\bar{x}}{\theta_0}\right) e^{-\frac{\bar{x}}{\theta_0}}$.

Note that

$$\frac{d}{dy} f(y) = e^{-y} - ye^{-y} = (1-y)e^{-y}.$$

The function $f(y)$ is thus decreasing for $y > 1$ and returning to the problem at hand we also have that $\left(\frac{\bar{x}}{\theta_0}\right) e^{-\frac{\bar{x}}{\theta_0}}$ is decreasing for $\left(\frac{\bar{x}}{\theta_0}\right) > 1$ (our case of interest, i.e. the case when $\lambda(\boldsymbol{x}) < 1$). *Low* values of $f\left(\frac{\bar{x}}{\theta_0}\right) = \left(\frac{\bar{x}}{\theta_0}\right) e^{-\frac{\bar{x}}{\theta_0}}$ are thus achieved by *high* values of $\frac{\bar{x}}{\theta_0}$, see Figure 1 below.

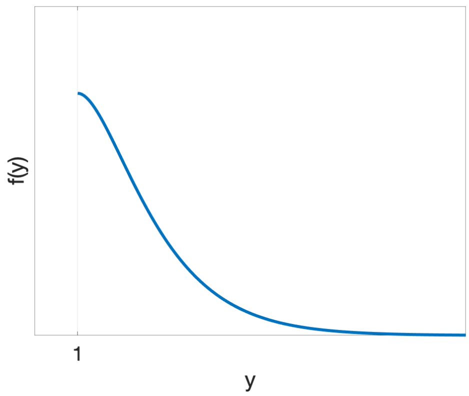

Figure 1: A visualization of the function $f(y)$.

We conclude that

$$\left(\frac{\bar{x}}{\theta_0}\right) e^{-\frac{\bar{x}}{\theta_0}} \leq k_1 e^{-1} = k_2 \quad \Rightarrow \quad \frac{\bar{x}}{\theta_0} \geq k_3 \quad \Rightarrow \quad \frac{2n\bar{x}}{\theta_0} \geq 2nk_3 = c,$$

where k_3 and c are constant to be determined to control the Type I error. Under the null hypothesis we have $\frac{2n\bar{X}}{\theta_0} \sim \chi^2(2n)$, therefore

$$\mathbb{P}\left(\frac{2n\bar{X}}{\theta_0} \geq c \,\middle|\, \theta_0\right) = \alpha \quad \Rightarrow \quad c = \chi^2_{1-\alpha}(2n).$$

For typical sizes, the GLR test of size α has critical region

$$C^* = \left\{x_1, \ldots, x_n \,\middle|\, \frac{2n\bar{x}}{\theta_0} \geq \chi^2_{1-\alpha}(2n)\right\}.$$

Exercise 29

We have $X_1, \ldots, X_n \sim \mathrm{UNIF}(0, \theta)$. The joint pdf is

$$f(\boldsymbol{x}; \theta) = \prod_{i=1}^{n} \frac{1}{\theta} \mathbb{1}\{x_i \leq \theta\} = \theta^{-n} \mathbb{1}\left\{\max_{i=1,\ldots,n} x_i \leq \theta\right\}.$$

If the null is true, then $\theta = \theta_0$, whereas the unrestricted ML estimate is $\hat{\theta} = \max_{1,\ldots,n} x_i$. We have

$$\lambda(\boldsymbol{x}) = \frac{\max_{\theta \in \Omega_0} f(\boldsymbol{x}; \theta)}{\max_{\theta \in \Omega} f(\boldsymbol{x}; \theta)} = \frac{f(\boldsymbol{x}; \theta_0)}{f(\boldsymbol{x}; \hat{\theta})} = \left(\frac{\max_{1,\ldots,n} x_i}{\theta_0}\right)^n \mathbb{1}\left\{\max_{i=1,\ldots,n} x_i \leq \theta_0\right\}.$$

We should reject the null hypothesis when $\lambda(\boldsymbol{x})$ is small. The most extreme situation occurs when $\lambda(\boldsymbol{x}) = 0$ because $\max_{i=1,\ldots,n} x_i$ exceeds θ_0. If this happens, then we know for sure that we should reject the null hypothesis because the event $\{\max_{i=1,\ldots,n} x_i > \theta_0\}$ cannot occur if $X_1, \ldots, X_n \sim \text{UNIF}(0, \theta_0)$. Actually, there is not really any reason to conduct a hypothesis test because we are certain that our null hypothesis $H_0 : \theta = \theta_0$ is false as soon as we observe a maximum outcome larger than θ_0. So let us rule out this scenario, and continue to see what is happening under H_0.

Under H_0, we have $X_1, \ldots, X_n \sim \text{UNIF}(0, \theta_0)$ and we must have $\mathbb{1}\{\max_{i=1,\ldots,n} X_i \leq \theta_0\} = 1$ with probability one. Rejection for small values of $\lambda(\boldsymbol{x})$ is thus equivalent to rejecting for small values of $\max_{1,\ldots,n} x_i$. Denoting the critical value by c, we must have

$$\mathbb{P}\left(\max_{i=1,\ldots,n} X_i \leq c \,\middle|\, \theta_0 \right) = \mathbb{P}\left(X_1 \leq c, \ldots, X_n \leq c \,|\, \theta_0 \right) = \Big[\mathbb{P}(X_1 \leq c \,|\, \theta_0) \Big]^n = \left(\frac{c}{\theta_0} \right)^n = \alpha.$$

to control for the probability of a Type I error. We conclude that $c = \theta_0\, \alpha^{1/n}$. The GLR test of size α has critical region

$$C^* = \left\{ x_1, \ldots, x_n \,\middle|\, \max_{i=1,\ldots,n} x_i \leq \theta_0\, \alpha^{1/n} \right\}.$$

Exercise 31

The joint pdf of the sample is $f(\boldsymbol{x}; \theta) = \prod_{i=1}^n \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^n x_i\right)^{\theta-1}$. Under H_0 we have only a single parameter value. It remains to compute the unrestricted estimator. The likelihood is $L(\theta) = \theta^n \left(\prod_{i=1}^n x_i\right)^{\theta-1}$ and log-likelihood is

$$\ln L(\theta) = n \ln(\theta) + (\theta - 1) \sum_{i=1}^n \ln(x_i).$$

The first and second derivative of the log-likelihood with respect to θ are

$$\begin{aligned} \frac{d}{d\theta} \ln L(\theta) &= \frac{n}{\theta} + \sum_{i=1}^n \ln(x_i), \\ \frac{d^2}{d\theta^2} \ln L(\theta) &= -\frac{n}{\theta^2} < 0, \qquad \text{for all } \theta. \end{aligned}$$

We obtain $\hat{\theta} = -\frac{n}{\sum_{i=1}^n \ln(x_i)}$ (the second order condition is automatically fulfilled). The GLR evaluates to

$$\lambda(\boldsymbol{x}) = \frac{\max_{\theta \in \Omega_0} f(\boldsymbol{x}; \theta)}{\max_{\theta \in \Omega} f(\boldsymbol{x}; \theta)} = \frac{f(\boldsymbol{x}; \theta_0)}{f(\boldsymbol{x}; \hat{\theta})} = \frac{\theta_0^n \left(\prod_{i=1}^n x_i\right)^{\theta_0 - 1}}{(\hat{\theta})^n \left(\prod_{i=1}^n x_i\right)^{\hat{\theta} - 1}} = \left(\frac{\theta_0}{\hat{\theta}} \right)^n \left(\prod_{i=1}^n x_i \right)^{\theta_0 - \hat{\theta}},$$

and we can additionally compute

$$\begin{aligned} -2 \ln\big(\lambda(\boldsymbol{x})\big) &= -2n \ln\left(\frac{\theta_0}{\hat{\theta}} \right) - 2(\theta_0 - \hat{\theta}) \sum_{i=1}^n \ln(x_i) \\ &= -2n \ln\left(\frac{\theta_0}{\hat{\theta}} \right) + 2n \left(\frac{\theta_0 - \hat{\theta}}{\hat{\theta}} \right), \end{aligned}$$

where we have used the definition of $\hat{\theta}$ to replace $\sum_{i=1}^{n} \ln(x_i)$. According to Equation (12.8.3), an approximate size α test is to reject H_0 if

$$-2\ln\big(\lambda(\boldsymbol{x})\big) = -2n\ln\left(\frac{\theta_0}{\hat{\theta}}\right) + 2n\left(\frac{\theta_0 - \hat{\theta}}{\hat{\theta}}\right) \geq \chi^2_{1-\alpha}(1).$$