Solutions to Selected Exercises from Chapter 9
Bain & Engelhardt - Second Edition

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Exercise 1

(a) First population moment can be calculated as

$$\mathbb{E}(X) = \int_0^1 x\theta x^{\theta-1}dx = \int_0^1 \theta x^{\theta}dx = \frac{\theta}{\theta+1}x^{\theta+1}\Big|_0^1 = \frac{\theta}{\theta+1}.$$

Equate it to the first sample moment and solve the equation to obtain the MME $\tilde{\theta}$:

$$\frac{\tilde{\theta}}{\tilde{\theta}+1} = \bar{X} \qquad \Rightarrow \qquad \tilde{\theta} = \frac{\bar{X}}{1-\bar{X}}.$$

(b) We again calculate $\mathbb{E}(X)$. The calculation shows

$$\mathbb{E}(X) = \int_1^{\infty} x(\theta+1)x^{-\theta-2}dx = \int_1^{\infty} (\theta+1)x^{-\theta-1}dx = -\frac{\theta+1}{\theta}x^{-\theta}\Big|_1^{\infty} = \frac{\theta+1}{\theta}$$

Equate it to the first sample moment and solve the equation to obtain the MME $\tilde{\theta}$:

$$\frac{\tilde{\theta}+1}{\tilde{\theta}} = \bar{X} \qquad \Rightarrow \qquad \tilde{\theta} = \frac{1}{\bar{X}-1}.$$

(c) The pdf corresponds to a GAM$(1/\theta, 2)$ distribution since $f(x;\theta) = \theta^2 x e^{-\theta x} = \frac{1}{(1/\theta)^2\Gamma(2)} x e^{-\frac{x}{1/\theta}}$. We can find the population moment $\mathbb{E}(X) = \frac{2}{\theta}$ from Table B.2. Equate it to the first sample moment and solve the equation to obtain the MME $\tilde{\theta}$:

$$\frac{2}{\tilde{\theta}} = \bar{X} \qquad \Rightarrow \qquad \tilde{\theta} = \frac{2}{\bar{X}}.$$

Exercise 3

(a) The likelihood function is $L(\theta) = \prod_{i=1}^n f(x_i;\theta) = \theta^n \left(\prod_{i=1}^n x_i\right)^{\theta-1}$ and the associated log-likelihood is

$$\ln L(\theta) = n\ln\theta + (\theta-1)\sum_{i=1}^n \ln(x_i).$$

The first and second derivative of the log-likelihood are:

$$\frac{d}{d\theta}\ln L(\theta) = \frac{n}{\theta} + \sum_{i=1}^n \ln(x_i), \qquad \frac{d^2}{d\theta^2}\ln L(\theta) = -\frac{n}{\theta^2} < 0.$$

The second derivative is negative for all values of θ. We can thus solve the first order condition to find the estimator $\hat{\theta} = -\frac{n}{\sum_{i=1}^n \ln(X_i)}$.

(b) The likelihood function is $L(\theta) = \prod_{i=1}^n f(x_i;\theta) = (\theta+1)^n \left(\prod_{i=1}^n x_i\right)^{-\theta-2}$ and the associated log-likelihood is

$$\ln L(\theta) = n\ln(\theta+1) - (\theta+2)\sum_{i=1}^n \ln(x_i).$$

The first and second derivative of the log-likelihood with respect to θ are

$$\frac{d}{d\theta}\ln L(\theta) = \frac{n}{\theta+1} - \sum_{i=1}^n \ln(x_i), \qquad \frac{d^2}{d\theta^2}\ln L(\theta) = -\frac{n}{(\theta+1)^2} < 0.$$

The second derivative is negative for all values of θ. We can thus solve the first order condition to find the estimator $\hat{\theta} = \frac{n}{\sum_{i=1}^n \ln(X_i)} - 1$.

(c) We have the likelihood function $L(\theta) = \prod_{i=1}^n f(x_i;\theta) = \theta^{2n}\left(\prod_{i=1}^n x_i\right) e^{-\theta\sum_{i=1}^n x_i}$ and log-likelihood function

$$\ln L(\theta) = 2n\ln(\theta) + \sum_{i=1}^n \ln(x_i) - \theta n\bar{x},$$

where we used $\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i$. The first derivative is $\frac{d}{d\theta}\ln L(\theta) = \frac{2n}{\theta} - n\bar{x}$. Solving the first order condition, i.e. $\frac{2n}{\theta} - \sum_{i=1}^n x_i = 0$, gives the candidate solution $\hat{\theta} = 2/\bar{x}$. For the second derivative we have

$$\frac{d^2}{d\theta^2}\ln L(\theta) = -\frac{2n}{\theta^2} < 0,$$

for all θ. We conclude that the maximum likelihood estimator is $\hat{\theta} = 2/\bar{X}$.

Exercise 5

For the given pdf, the likelihood function equals

$$L(\theta) = \prod_{i=1}^n f(x_i;\theta) = 2^n\theta^{2n}\left(\prod_{i=1}^n x_i\right)^{-3}, \qquad 0 < \theta \le x_{1:n} = \min_{1\le i\le n} x_i.$$

This likelihood is strictly monotonically increasing in θ and we would correspondingly like to take θ as large as possible. However, due to the restriction $0 < \theta \le x_{1:n}$, the likelihood will be zero whenever θ exceeds $x_{1:n}$. It follows that the ML estimator is $\hat{\theta} = X_{1:n}$.

Exercise 7

All the quantities in exercises (a)-(c) are transformations of p. We will thus first derive the MLE for the parameter p and subsequently use the invariance property. First the derivation of the likelihood function

$$L(p) = \prod_{i=1}^n f(x_i;p) = p^n(1-p)^{\sum_{i=1}^n x_i - n} = p^n(1-p)^{n(\bar{x}-1)},$$

(using the definition of $\bar{x}$) and log-likelihood function

$$\ln L(p) = n\ln(p) + n(\bar{x}-1)\ln(1-p).$$

The first derivative of the log-likelihood function is

$$\frac{d}{dp}\ln L(p) = \frac{n}{p} - \frac{n(\bar{x}-1)}{1-p} = 0.$$

We can find a candidate solution for the MLE by setting this first derivative equal to zero, that is

$$\frac{n}{\hat{p}} - \frac{n(\bar{x}-1)}{1-\hat{p}} = 0 \qquad \Rightarrow \qquad \hat{p} = \frac{1}{\bar{x}}.$$

It remains to verify whether our candidate solution indeed gives a maximum. For this we should show that $\frac{d^2}{dp^2}\ln L(p)\Big|_{p=\hat{p}} < 0$. By differentiation we find

$$\frac{d^2}{dp^2}\ln L(p)\Bigg|_{p=\hat{p}} = -\frac{n}{p^2} - \frac{n(\bar{x}-1)}{(1-p)^2}\Bigg|_{p=\hat{p}} = -n\frac{\bar{x}^3}{\bar{x}-1} < 0$$

since $x \in \{1, 2, 3, \ldots\}$ hence $\bar{x} > 1$ (we rule out the case when we observe a sample of only ones because this gives no information about the parameter). The ML estimator is thus $\hat{p} = \frac{1}{\bar{X}}$. The subquestions are now quick to answer using the Invariance Property of MLEs (Theorem 9.2.2 on page 298 of B&E).

(a) $\tau(p) = \mathbb{E}(X) = \frac{1}{p}$, hence the MLE is $\tau(\hat{p}) = \frac{1}{\hat{p}} = \bar{X}$

(b) $\tau(p) = \mathbb{V}\text{ar}(X) = \frac{1-p}{p^2}$, hence the MLE is $\tau(\hat{p}) = \frac{1-\hat{p}}{\hat{p}^2} = \bar{X}(\bar{X}-1)$

(c) $\tau(p) = \mathbb{P}(X > k) = (1-p)^k$, hence the MLE is $\tau(\hat{p}) = (1-\hat{p})^k = \left(1 - \frac{1}{\bar{X}}\right)^k$ for arbitrary $k = 1, 2, \ldots$

Exercise 15

(a) If $X \sim \text{BIN}(n, p)$, then $\mathbb{E}(X) = np$ and $\mathbb{V}\text{ar}(X) = np(1-p)$ (see Table B.2). We have

$$\begin{aligned}
\mathbb{E}\left[c\hat{p}(1-\hat{p})\right] &= \mathbb{E}\left[c\frac{X}{n}\left(1-\frac{X}{n}\right)\right] = \frac{c}{n}\,\mathbb{E}\left[X - \frac{X^2}{n}\right] = \frac{c}{n}\left[\mathbb{E}(X) - \frac{\mathbb{E}(X^2)}{n}\right] \\
&= \frac{c}{n}\left(\mathbb{E}(X) - \frac{\mathbb{V}\text{ar}(X) + (\mathbb{E}(X))^2}{n}\right) = \frac{c}{n}\left(np - \frac{np(1-p) + (np)^2}{n}\right) \\
&= \frac{c}{n}\left(np - p(1-p) - np^2\right) = \frac{c}{n}\Big(np(1-p) - p(1-p)\Big) = c\frac{n-1}{n}p(1-p).
\end{aligned}$$

$\mathbb{E}\left[c\hat{p}(1-\hat{p})\right] = p(1-p)$ will hold when $c = \frac{n}{n-1}$.

(b) Note that $\mathbb{V}\text{ar}(X) = np(1-p)$. In view of the previous exercise we obtain the unbiased estimator $\frac{n^2}{n-1}\hat{p}(1-\hat{p})$.

(c) We now have a random sample $X_1, \ldots, X_N \sim \text{BIN}(n, p)$. The fact that $\mathbb{E}(X) = np$ suggest the estimator $\hat{p}^* = \frac{1}{nN}\sum_{i=1}^{N} X_i$. The following calculation shows that this is indeed an unbiased estimator:

$$\mathbb{E}(\hat{p}^*) = \mathbb{E}\left(\frac{1}{nN}\sum_{i=1}^{N} X_i\right) = \frac{1}{nN}\sum_{i=1}^{N}\mathbb{E}(X_i) = \frac{1}{nN}N(np) = p.$$

Similarly, an unbiased estimator for $\mathbb{V}\text{ar}(X) = np(1-p)$ is easily constructed using the answer to part (a). Defining the estimator as $\widehat{\mathbb{V}\text{ar}(X)} = \frac{1}{N}\sum_{i=1}^{N} \frac{n^2}{n-1}\frac{X_i}{n}(1-\frac{X_i}{n})$, we have

$$\mathbb{E}\left(\widehat{\mathbb{V}\text{ar}(X)}\right) = \mathbb{E}\left(\frac{1}{N}\sum_{i=1}^{N}\frac{n^2}{n-1}\frac{X_i}{n}\left(1-\frac{X_i}{n}\right)\right) = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\left(\frac{n^2}{n-1}\frac{X_i}{n}\left(1-\frac{X_i}{n}\right)\right) = \mathbb{V}\text{ar}(X).$$

Exercise 17

(a) Since $X \sim \text{UNIF}(\theta-1, \theta+1)$, we have $\mathbb{E}(X) = \frac{\theta-1+\theta+1}{2} = \theta$ (see Table B.2) and also $\mathbb{E}(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i) = \theta$. $\bar{X}$ is thus an unbiased estimator for θ.

(b) The pdf of the uniform distribution $\text{UNIF}(\theta-1,\theta+1)$ is

$$f(x;\theta) = \frac{1}{2} \qquad \theta-1 < x < \theta+1$$

and the CDF is

$$F(x;\theta) = \begin{cases} 0 & x \le \theta - 1 \\ \frac{x-\theta+1}{2} & \theta-1 < x < \theta+1 \\ 1 & x \ge \theta+1 \end{cases}$$

(see page 109). Using Theorem 6.5.2 (page 217 of B&E), we see that the pdfs of the order statistics $X_{1:n}$ and $X_{n:n}$ are

$$\begin{aligned} g_1(x) &= n(1-F(x))^{n-1}f(x) = \frac{n}{2^n}(\theta+1-x)^{n-1} && \theta-1 < x < \theta+1 \\ g_n(x) &= n(F(x))^{n-1}f(x) = \frac{n}{2^n}(x-\theta+1)^{n-1} && \theta-1 < x < \theta+1, \end{aligned}$$

respectively. First, we calculate $\mathbb{E}(X_{1:n})$:

$$\begin{aligned} \mathbb{E}(X_{1:n}) &= \int_{\theta-1}^{\theta+1} x\frac{n}{2^n}(\theta+1-x)^{n-1}dx = \int_0^2 (\theta+1-y)\frac{n}{2^n}y^{n-1}dy \\ &= (\theta+1) - \int_0^2 \frac{n}{2^n}y^n dy = (\theta+1) - \frac{2n}{n+1} = (\theta-1) + \frac{2}{n+1}, \end{aligned}$$

by changing the integration variable to $y = \theta+1-x$. In other words, $X_{1:n}$ is on average $\frac{2}{n+1}$ higher than the lower bound $\theta-1$. Second, for $\mathbb{E}(X_{n:n})$, we have

$$\begin{aligned} \mathbb{E}(X_{n:n}) &= \int_{\theta-1}^{\theta+1} x\frac{n}{2^n}(x-\theta+1)^{n-1}dx = \int_0^2 (z+\theta-1)\frac{n}{2^n}z^{n-1}dz \\ &= (\theta-1) + \int_0^2 \frac{n}{2^n}z^n dz = (\theta-1) + \frac{2n}{n+1} = (\theta+1) - \frac{2}{n+1}, \end{aligned}$$

after changing the integration variable to $z = x-\theta+1$. We see that $X_{n:n}$ is lower than the upper bound $\theta+1$ by $\frac{2}{n+1}$ (the same quantity as before). Finally, by linearity of the expectation, we have

$$\mathbb{E}\left(\frac{X_{1:n}+X_{n:n}}{2}\right) = \frac{\mathbb{E}(X_{1:n})+\mathbb{E}(X_{n:n})}{2} = \frac{(\theta-1)+\frac{2}{n+1}+(\theta+1)-\frac{2}{n+1}}{2} = \theta,$$

and we see that the "midrange" is indeed an unbiased estimator for θ.

Exercise 21

(a) If $X \sim \text{BIN}(1, p)$, then $\mathbb{E}(X) = p$ and $\mathbb{V}\text{ar}(X) = p(1-p)$ (see Table B.2). For the numerator of the CRLB, we have $\tau(p) = p$ and thus $\tau'(p) = 1$. The following calculations can be used to evaluated the expectation in the denominator:

$$\begin{aligned} f(x;p) &= p^x(1-p)^{1-x} \\ \ln f(x;p) &= x\ln p + (1-x)\ln(1-p) \\ \frac{\partial}{\partial p}\ln f(x;p) &= \frac{x}{p} - \frac{1-x}{1-p} = \frac{x-p}{p(1-p)} \\ \mathbb{E}\left(\frac{\partial}{\partial p}\ln f(X;p)\right)^2 &= \mathbb{E}\left(\frac{X-p}{p(1-p)}\right)^2 = \frac{\mathbb{E}(X-p)^2}{p^2(1-p)^2} = \frac{\mathbb{V}\text{ar}(X)}{p^2(1-p)^2} = \frac{1}{p(1-p)} \end{aligned}$$

The CRLB is now obtained as

$$\frac{[\tau'(p)]^2}{n\,\mathbb{E}\left(\frac{\partial}{\partial p}\ln f(X;p)\right)^2} = \frac{1}{\frac{n}{p(1-p)}} = \frac{p(1-p)}{n}.$$

(b) Only the numerator of the CRLB will change. We now have $\tau(p) = p(1-p)$ such that $\tau'(p) = 1 - 2p$. The CRLB is

$$\frac{[\tau'(p)]^2}{n\,\mathbb{E}\left(\frac{\partial}{\partial p}\ln f(X;p)\right)^2} = \frac{(1-2p)^2}{\frac{n}{p(1-p)}} = \frac{p(1-p)(1-2p)^2}{n}.$$

(c) Looking at your answer for part (a) you should recognize that the CRLB coincides with $\mathbb{V}\text{ar}(X)/n$. As an educated guess we therefore try $\hat{p} = \bar{X}$. First, from $\mathbb{E}(X) = p$, we see that

$$\mathbb{E}(\hat{p}) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i) = p$$

and conclude that $\bar{X}$ is an unbiased estimator for p. The variance from this estimator, i.e.

$$\mathbb{V}\text{ar}(\hat{p}) = \mathbb{V}\text{ar}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{V}\text{ar}(X_i) = \frac{p(1-p)}{n}$$

is seen to attain the CRLB. We conclude that $\hat{p} = \bar{X}$ is an UMVUE of p.

Exercise 22

(a) For the numerator of the CRLB, we find $\tau(\mu) = \mu$ and $\tau'(\mu) = 1$. The next intermediate steps can be used to evaluated the expectation in the denominator:

$$\begin{aligned} f(x;\mu) &= \frac{1}{\sqrt{2\pi}3}e^{-\frac{(x-\mu)^2}{18}} \\ \ln f(x;\mu) &= -\ln(\sqrt{2\pi}3) - \frac{(x-\mu)^2}{18} \\ \frac{\partial}{\partial \mu}\ln f(x;\mu) &= \frac{x-\mu}{9} \\ \mathbb{E}\left(\frac{\partial}{\partial \mu}\ln f(X;\mu)\right)^2 &= \mathbb{E}\left(\frac{X-\mu}{9}\right)^2 = \frac{\mathbb{E}(X-\mu)^2}{81} = \frac{\mathbb{V}\text{ar}(X)}{81} = \frac{1}{9} \end{aligned}$$

The CRLB is now obtained as

$$\frac{[\tau'(\mu)]^2}{n\,\mathbb{E}\left(\frac{\partial}{\partial\mu}\ln f(X;\mu)\right)^2} = \frac{1}{\frac{n}{9}} = \frac{9}{n}.$$

(b) The expectation and variance of $\hat{\mu}$ are

$$\mathbb{E}(\hat{\mu}) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \mu,$$
$$\mathbb{V}\text{ar}(\hat{\mu}) = \mathbb{V}\text{ar}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} \mathbb{V}\text{ar}(X_i) = \frac{9}{n}.$$

The final expression for the expectation shows that $\hat{\mu}$ is an unbiased estimator for μ. Since the variance of $\hat{\mu}$ also attains the CRLB, we can conclude that $\hat{\mu}$ is an UMVUE for μ.

(c) The 95% percentile of $X \sim \mathrm{N}(\mu, 9)$ can be written as $\tau(\mu) = \mu + 3z_{0.95}$, since $Z = \frac{X-\mu}{3} \sim \mathrm{N}(0,1)$ (remember that $z_{0.95}$ denotes the 95% percentile of the standard normal distribution). By the invariance property $\tau(\hat{\mu}) = \bar{X} + 3z_{0.95}$ is the MLE of $\tau(\mu)$. In addition, $\tau'(\mu) = 1$ implies that the CRLB remains $\frac{9}{n}$. Since

$$\mathbb{E}\left(\tau(\hat{\mu})\right) = 3z_{0.95} + \mathbb{E}(\bar{X}) = 3z_{0.95} + \mu = \tau(\mu)$$

and

$$\mathbb{V}\text{ar}\left(\tau(\hat{\mu})\right) = \mathbb{V}\text{ar}(\bar{X}) = \frac{9}{n},$$

it follows that $\tau(\hat{\mu}) = 3z_{0.95} + \bar{X}$ is an UMVUE of $\tau(\mu)$.

Exercise 23

(a) We first have to derive the MLE for θ. We proceed with the usual steps. The likelihood function is $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = (2\pi\theta)^{-n/2}\exp\left(-\frac{\sum_{i=1}^{n} x_i^2}{2\theta}\right)$ and therefore

$$\ln L(\theta) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\theta) - \frac{\sum_{i=1}^{n} x_i^2}{2\theta}.$$

We subsequently compute the first two derivatives as

$$\frac{d}{d\theta}\ln L(\theta) = -\frac{n}{2\theta} + \frac{\sum_{i=1}^{n} x_i^2}{2\theta^2}$$
$$\frac{d^2}{d\theta^2}\ln L(\theta) = \frac{n}{2\theta^2} - \frac{\sum_{i=1}^{n} x_i^2}{\theta^3}.$$

If we equate the first derivate to zero and solve for the estimator, then we find $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i^2$. The second order condition is fulfilled because

$$\left.\frac{d^2}{d\theta^2}\ln L(\theta)\right|_{\theta=\hat{\theta}} = \frac{n}{2\hat{\theta}^2} - \frac{n\hat{\theta}}{\hat{\theta}^3} = -\frac{n}{2\hat{\theta}^2} < 0.$$

The MLE for θ is thus $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. From

$$\mathbb{E}(\hat{\theta}) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i^2) = \frac{1}{n}\sum_{i=1}^{n}\mathrm{Var}(X_i) = \theta,$$

we see that this estimator is unbiased for θ.

(b) Starting from $f(X;\theta) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{x^2}{2\theta}\right)$, we find

$$\ln f(X;\theta) = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(\theta) - \frac{X^2}{2\theta},$$

and by steps similar to those in part (a), the second derivatie w.r.t. θ becomes

$$\frac{\partial^2}{\partial\theta^2}\ln f(X;\theta) = \frac{1}{2\theta^2} - \frac{X^2}{\theta^3}.$$

Note that $X/\sqrt{\theta} \sim \mathrm{N}(0,1)$, and thus $\frac{X^2}{\theta} \sim \chi^2(1)$. Using the latter result, we have $-\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\ln f(X;\theta)\right] = \frac{1}{\theta^2}\mathbb{E}\left(\frac{X^2}{\theta}\right) - \frac{1}{2\theta^2} = \frac{1}{2\theta^2}$ and a CRLB evaluating to

$$\left(-n\,\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\ln f(X;\theta)\right]\right)^{-1} = \frac{2\theta^2}{n}.$$

The variance of $\hat{\theta}$ is

$$\mathbb{V}\mathrm{ar}(\hat{\theta}) = \mathbb{V}\mathrm{ar}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \mathbb{V}\mathrm{ar}\left(\frac{\theta}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\sqrt{\theta}}\right)^2\right) = \frac{\theta^2}{n^2}\mathbb{V}\mathrm{ar}(Y_n) = \frac{\theta^2}{n^2}(2n) = \frac{2\theta^2}{n},$$

where we made use of the random variable $Y_n = \frac{1}{\theta}\sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n}\left(\frac{X_i}{\sqrt{\theta}}\right)^2$. Since $X_i/\sqrt{\theta} \sim \mathrm{N}(0,1)$, we know that this Y_n is the sum of squared (and independent) standard normal random variables. Therefore, $Y_n \sim \chi^2(n)$ and $\mathbb{V}\mathrm{ar}(Y_n) = 2n$. The estimator $\hat{\theta}$ is thus unbiased (see (a)) and its variance attains the CRLB. We conclude that $\hat{\theta}$ is an UMVUE for θ.

Exercise 26

(a) The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \frac{1}{\theta^n}, \qquad 0 < x_{1:n} = \min_{1\le i\le n} x_i, \qquad x_{n:n} = \max_{1\le i\le n} x_i \le \theta.$$

$L(\theta)$ is strictly decreasing in θ, so $L(\theta)$ is maximal if θ is minimal. However, the restriction $x_{n:n} \le \theta$ implies that we should not decrease θ below the value $x_{n:n}$ (otherwise the likelihood would become zero). It follows that $\hat{\theta} = X_{n:n}$.

(b) The given pdf corresponds to the UNIF$(0,\theta)$ distribution. If the random variable X his this particular uniform distribution, then $\mathbb{E}(X) = \theta/2$. The estimator follows from:

$$\bar{X} = \frac{\tilde{\theta}}{2} \qquad \Rightarrow \qquad \tilde{\theta} = 2\bar{X}.$$

(c) The CDF related to the UNIF$(0,\theta)$ distribution is

$$F(x;\theta) = \begin{cases} 0 & x \le 0 \\ \frac{x}{\theta} & 0 < x \le \theta \\ 1 & x > \theta. \end{cases}$$

The pdf of the order statistic $X_{n:n}$, cf. Theorem 6.5.2 (page 217 of B&E), is thus equal to

$$g_n(x) = n(F(x))^{n-1} f(x) = \frac{n}{\theta^n} x^{n-1}, \qquad 0 < x \le \theta,$$

(and zero elsewhere). We can now find $\mathbb{E}(X_{n:n})$ by integration,

$$\mathbb{E}(\hat{\theta}) = \mathbb{E}(X_{n:n}) = \int_0^\theta \frac{n}{\theta^n} x^n dx = \frac{n}{\theta^n} \left. \frac{x^{n+1}}{n+1} \right|_0^\theta = \frac{n}{n+1}\theta \neq \theta,$$

which reveals that the MLE $\hat{\theta}$ is biased.

(d) Use $\mathbb{E}(X) = \frac{\theta}{2}$ and linearity of the expectation operator to find

$$\mathbb{E}(\tilde{\theta}) = \mathbb{E}(2\bar{X}) = \mathbb{E}\left(\frac{2}{n}\sum_{i=1}^n X_i\right) = \frac{2}{n}\sum_{i=1}^n \mathbb{E}(X_i) = \theta.$$

We now see that MME $\tilde{\theta}$ is unbiased.

(e) Let us start with the MLE $\hat{\theta}$. We have $\text{MSE}(\hat{\theta}) = \mathbb{E}\left(X_{n:n} - \theta\right)^2 = \mathbb{E}(X_{n:n}^2) - 2\theta\,\mathbb{E}(X_{n:n}) + \theta^2$. We have computed $\mathbb{E}(X_{n:n})$ in part (c), so it remains to compute the second moment of the estimator. The calculation shows

$$\mathbb{E}(X_{n:n}^2) = \int_0^\theta \frac{n}{\theta^n} x^{n+1} dx = \frac{n}{\theta^n} \left. \frac{x^{n+2}}{n+2} \right|_0^\theta = \frac{n}{n+2}\theta^2.$$

We can now evaluate the expression from before. We find

$$\text{MSE}(\hat{\theta}) = \frac{n}{n+2}\theta^2 - 2\frac{n}{n+1}\theta^2 + \theta^2 = \frac{2\theta^2}{(n+1)(n+2)}.$$

We continue with the method of moments estimator $\tilde{\theta}$. This estimator was unbiased such that $\text{MSE}(\tilde{\theta}) = \mathbb{V}\text{ar}(\tilde{\theta})$. Since $\text{Var}(X) = \frac{\theta^2}{12}$, we can easily compute this variance by exploiting the standard properties of variances, namely

$$\text{MSE}(\tilde{\theta}) = \mathbb{V}\text{ar}(\tilde{\theta}) = \mathbb{V}\text{ar}(2\bar{X}) = \mathbb{V}\text{ar}\left(\frac{2}{n}\sum_{i=1}^n X_i\right) = \frac{4}{n^2}\sum_{i=1}^n \mathbb{V}\text{ar}(X_i) = \frac{\theta^2}{3n}.$$

The MSE of both estimator scales linearly with θ^2 (which is expected because this is the scale parameter). More interesting is the behavior as a function of n. The MLE will have a smaller (or equal) MSE, that is $\text{MSE}(\hat{\theta}) \le \text{MSE}(\tilde{\theta})$, when

$$\frac{2\theta^2}{(n+1)(n+2)} \le \frac{\theta^2}{3n} \qquad \Rightarrow \qquad n^2 - 3n + 2 = (n-1)(n-2) \ge 0.$$

The parabolic function $f(x) = (x-1)(x-2)$ intersect the x-axis in the points $x = 1$ and $x = 2$. We therefore conclude that the MLE for θ has an MSE that is never higher than the MSE of the MME (for any sample size $n = 1, 2, \ldots$).

Exercise 31
The MSEs for $\hat{\theta}$ and $\tilde{\theta}$ were computed in Exercise 26. According to Definition 9.4.2 (page 312 in B&E) we only have to take the limit $n \to \infty$.

(a) $\lim_{n\to\infty} \text{MSE}(\hat{\theta}_n) = \lim_{n\to\infty} \frac{2\theta^2}{(n+1)(n+2)} = 0$, hence $\hat{\theta}_n = X_{n:n}$ is MSE consistent for θ.

(b) $\lim_{n\to\infty} \text{MSE}(\tilde{\theta}_n) = \lim_{n\to\infty} \frac{\theta^2}{3n} = 0$, hence $\tilde{\theta}_n = 2\bar{X}_n$ is MSE consistent for θ.

Exercise 32
In Exercise 5 we have seen that $\hat{\theta}_n = X_{1:n}$. From $\int_\theta^x 2\theta^2 t^{-3} dt = -\theta^2 t^{-2}\big|_\theta^x = 1 - \theta^2 x^{-2}$, we find the following CDF for $\hat{\theta}$:

$$F(x;\theta) = \begin{cases} 0 & x \le \theta \\ 1 - \theta^2 x^{-2} & \theta < x. \end{cases}$$

The related pdf for the estimator $\hat{\theta}$ is[1]

$$g_1(x) = n(1 - F(x))^{n-1} f(x) = 2n\theta^{2n} x^{-2n-1} \qquad \theta \le x,$$

and zero otherwise. We can now compute the probability stated in Definition 9.4.1 (page 311 in B&E) explicitly:

$$\begin{aligned}
\mathbb{P}\left(|\hat{\theta}_n - \theta| < \varepsilon\right) = \mathbb{P}\left(X_{1:n} - \theta < \varepsilon\right) = \mathbb{P}\left(X_{1:n} < \theta + \varepsilon\right) &= \int_\theta^{\theta+\varepsilon} 2n\theta^{2n} x^{-2n-1} dx \\
= -\theta^{2n} x^{-2n}\big|_\theta^{\theta+\varepsilon} = 1 - \theta^{2n}(\theta+\varepsilon)^{-2n} &= 1 - \left(\frac{\theta}{\theta+\varepsilon}\right)^{2n}.
\end{aligned}$$

Since $0 < \left(\frac{\theta}{\theta+\varepsilon}\right) < 1$ for any $\epsilon > 0$, we obtain $\lim_{n\to\infty} \mathbb{P}\left(|\hat{\theta}_n - \theta| < \varepsilon\right) = 1$ thereby showing that the MLE $\hat{\theta}_n = X_{1:n}$ is (simply) consistent for θ.

[1] The CDF for $X_{1:n}$ is $\mathbb{P}(X_{1:n} \le x) = 1 - \mathbb{P}(X_{1:n} > x) = 1 - \mathbb{P}(X_1 > x, X_2 > x, \ldots, X_n > x) = 1 - \prod_{i=1}^n \mathbb{P}(X_i > x) = 1 - [1 - F(x)]^n$. By differentiation w.r.t. x we find $g_1(x) = n(1 - F(x))^{n-1} f(x)$.

Exercise 33

(a) If $X \sim \text{POI}(\mu)$, then $\mathbb{E}(X) = \mathbb{V}\text{ar}(X) = \mu$ (see Table B.2). For the numerator of the CRLB, $\tau(\mu) = \mu$ yields $\tau'(\mu) = 1$. The following calculations can be used to evaluated the expectation in the denominator:

$$\begin{aligned} f(x;\mu) &= \frac{e^{-\mu}\mu^x}{x!} \\ \ln f(x;\mu) &= -\mu + x\ln\mu - \ln(x!) \\ \frac{\partial}{\partial\mu}\ln f(x;\mu) &= -1 + \frac{x}{\mu} = \frac{x-\mu}{\mu} \\ \mathbb{E}\left(\frac{\partial}{\partial\mu}\ln f(X;p)\right)^2 &= \mathbb{E}\left(\frac{x-\mu}{\mu}\right)^2 = \frac{\mathbb{E}(X-\mu)^2}{\mu^2} = \frac{\mathbb{V}\text{ar}(X)}{\mu^2} = \frac{1}{\mu}. \end{aligned}$$

The CRLB is thus equal to

$$\frac{[\tau'(\mu)]^2}{n\,\mathbb{E}\left(\frac{\partial}{\partial\mu}\ln f(X;\mu)\right)^2} = \frac{1}{\frac{n}{\mu}} = \frac{\mu}{n}.$$

(b) The denominator of the CRLB remains unchanged. But now $\theta = \tau(\mu) = e^{-\mu}$, hence $\tau'(\mu) = -e^{-\mu}$. The new CRLB is thus

$$\frac{[\tau'(\mu)]^2}{n\,\mathbb{E}\left(\frac{\partial}{\partial\mu}\ln f(X;\mu)\right)^2} = \frac{(-e^{-\mu})^2}{\frac{n}{\mu}} = \frac{\mu e^{-2\mu}}{n}.$$

(c) The CRLB for μ has the form $\mathbb{V}\text{ar}(X)/n$. The sample mean is therefore a promising candidate for an UMVUE. We compute mean and variance of our candidate estimator $\hat{\mu} = \bar{X}$ and find

$$\begin{aligned} \mathbb{E}(\hat{\mu}) &= \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i) = \mu, \\ \mathbb{V}\text{ar}(\hat{\mu}) &= \mathbb{V}\text{ar}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{V}\text{ar}(X_i) = \frac{\mu}{n}. \end{aligned}$$

From $\mathbb{E}(\hat{\mu})$ we see that $\hat{\mu}$ is an unbiased estimator for μ and its variance attains the CRLB. We conclude that $\hat{\mu} = \bar{X}$ is an UMVUE for μ.

(d) We use the invariance property for the transformation $\theta = \tau(\mu) = e^{-\mu}$. The MLE for θ is thus $\hat{\theta} = \tau(\hat{\mu}) = e^{-\bar{X}}$.

(e) Let us define the new random variable $Y_n = \sum_{i=1}^{n} X_i$. It can by shown (using for instance the properties of moment generating functions) that $Y_n \sim \text{POI}(n\mu)$. We have

$$\mathbb{E}(\hat{\theta}) = \mathbb{E}\left(e^{-\bar{X}}\right) = \mathbb{E}\left(e^{-\frac{1}{n}Y_n}\right) = M_{Y_n}\left(-\frac{1}{n}\right) = e^{n\mu\left(e^{-\frac{1}{n}}-1\right)} = \theta^{n\left(1-e^{-\frac{1}{n}}\right)} \neq \theta,$$

thereby showing that $\hat{\theta}$ is not an unbiased estimator for θ.

(f) $\hat{\theta}$ is asymptotically unbiased for θ if $\lim_{n\to\infty} \mathbb{E}(\hat{\theta}) = \theta$. The latter expectation was already calculated in the previous part so it remains to take the limit. Using the rules of limits, we have

$$\lim_{n\to\infty} \mathbb{E}(\hat{\theta}) = \lim_{n\to\infty} \theta^{n\left(1-e^{-\frac{1}{n}}\right)} = \theta^{\lim_{n\to\infty} n\left(1-e^{-\frac{1}{n}}\right)}.$$

The limit in the exponent can be written as $\lim_{n\to\infty} \frac{1-e^{-1/n}}{1/n}$ which at first sight gives the indeterminate form $\frac{0}{0}$. As a possible solution one may realize that $1/n$ becomes small as $n \to \infty$ which motivates the use of a Taylor series for the exponential (that is $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$). This results in

$$\begin{aligned} n\left[1 - e^{-\frac{1}{n}}\right] &= n\left[1 - \left(1 - \frac{1}{n} + \frac{1}{2n^2} - \frac{1}{3!n^3} + \frac{1}{4!n^4} - \dots\right)\right] \\ &= 1 - \frac{1}{2n} + \frac{1}{3!n^2} - \frac{1}{4!n^3} + \dots \to 1 \text{ for } n \to \infty, \end{aligned}$$

and hence $\lim_{n\to\infty} \mathbb{E}(\hat{\theta}) = \lim_{n\to\infty} \theta^{n\left(1-e^{-\frac{1}{n}}\right)} = \theta$. $\hat{\theta}$ is asymptotically unbiased for θ.

(g) Using the previously defined Y_n, we have $\tilde{\theta} = \left(\frac{n-1}{n}\right)^{\sum_{i=1}^n X_i} = \left(\frac{n-1}{n}\right)^{Y_n}$. Then

$$\begin{aligned} \mathbb{E}(\tilde{\theta}) &= \mathbb{E}\left(\left(\frac{n-1}{n}\right)^{Y_n}\right) = \mathbb{E}\left(e^{Y_n \log\left(\frac{n-1}{n}\right)}\right) = M_{Y_n}\left(\log\left(\frac{n-1}{n}\right)\right) = e^{n\mu\left(e^{\log\left(\frac{n-1}{n}\right)}-1\right)} \\ &= e^{n\mu\left(\frac{n-1}{n}-1\right)} = e^{-\mu} = \theta, \end{aligned}$$

and we see that $\tilde{\theta}$ is indeed unbiased for θ.

(h) We will use $\mathbb{V}\mathrm{ar}(\tilde{\theta}) = \mathbb{E}\left(\tilde{\theta}^2\right) - \left[\mathbb{E}(\tilde{\theta})\right]^2$ for which we only need to compute $\mathbb{E}(\tilde{\theta}^2)$. We have

$$\begin{aligned} \mathbb{E}(\tilde{\theta}^2) &= \mathbb{E}\left(\left[\left(\frac{n-1}{n}\right)^{Y_n}\right]^2\right) = \mathbb{E}\left(e^{2\log\left(\frac{n-1}{n}\right)Y_n}\right) = M_{Y_n}\left(2\log\left(\frac{n-1}{n}\right)\right) \\ &= e^{n\mu\left(e^{2\log\left(\frac{n-1}{n}\right)}-1\right)} = e^{n\mu\left(\left(\frac{n-1}{n}\right)^2-1\right)} = e^{-\mu\left(2-\frac{1}{n}\right)}, \end{aligned}$$

and find

$$\mathbb{V}\mathrm{ar}(\tilde{\theta}) = \mathbb{E}\left(\tilde{\theta}^2\right) - \left[\mathbb{E}(\tilde{\theta})\right]^2 = e^{-\mu\left(2-\frac{1}{n}\right)} - [e^{-\mu}]^2 = e^{-2\mu+\frac{\mu}{n}} - e^{-2\mu} = e^{-2\mu}\left(e^{\frac{\mu}{n}} - 1\right).$$

We should compare this expression to the CRLB which was found in part (b), that is $\frac{\mu e^{-2\mu}}{n}$. Another application of the Taylor series for the exponential results in

$$\begin{aligned} \mathbb{V}\mathrm{ar}(\tilde{\theta}) &= e^{-2\mu}\left(e^{\frac{\mu}{n}} - 1\right) = e^{-2\mu}\left(\left(1 + \frac{\mu}{n} + \frac{\mu^2}{2n^2} + \frac{\mu^3}{3!n^3} + \dots\right) - 1\right) \\ &= \frac{\mu e^{-2\mu}}{n}\left(1 + \frac{\mu}{2n} + \frac{\mu^2}{3!n^2} + \dots\right). \end{aligned}$$

Note that the higher order terms like $\frac{\mu}{2n}$, $\frac{\mu^2}{3!n^2}$ et cetera will all positive be positive. The variance of $\tilde{\theta}$ is thus greater than the CRLB.

Exercise 34

(a) The likelihood and log-likelihood are $L(p) = \prod_{i=1}^{n} p(1-p)^{x_i} = p^n(1-p)^{n\bar{x}}$ and

$$\ln L(p) = n\ln(p) + n\bar{x}\ln(1-p),$$

respectively. The first and second derivative of this log-likelihood with respect to the parameter p are

$$\frac{d}{dp}\ln L(p) = \frac{n}{p} - \frac{n\bar{x}}{1-p} = \frac{n - np(1+\bar{x})}{p(1-p)}$$
$$\frac{d^2}{dp^2}\ln L(p) = -\frac{n}{p^2} - \frac{n\bar{x}}{(1-p)^2}.$$

Equating the first derivative to zero yields the candidate solution $\hat{p} = \frac{1}{1+\bar{x}}$. The following calculation shows that this indeed gives a maximum

$$\left.\frac{d^2}{dp^2}\ln L(p)\right|_{p=\hat{p}} = -n(1+\bar{x})^2 - \frac{n(1+\bar{x})^2}{\bar{x}},$$

where we used $1-\hat{p} = \frac{\bar{x}}{1+\bar{x}}$ and rule out the case where $\bar{x} = 0$ (this can happen with finite probability yet will not give us information regarding p). We conclude that the MLE for p is $\hat{p} = \frac{1}{1+\bar{X}}$.

(b) Apply the invariance property to $\theta = \tau(p) = \frac{1-p}{p}$. The MLE is obtained as

$$\hat{\theta} = \tau(\hat{p}) = \frac{1 - \frac{1}{1+\bar{X}}}{\frac{1}{1+\bar{X}}} = \bar{X}.$$

(c) If X has pdf $f(x;p) = p(1-p)^x$ for $x = 0, 1, \ldots$, then $Y = 1 + X$ has the GEO(p) distribution. By linearity of the expectation we get $\mathbb{E}(X) = \mathbb{E}(Y) - 1 = \frac{1}{p} - 1$ (see Table B.2). This expectation will be needed later on. For the numerator of the CRLB, $\tau(p) = \frac{1-p}{p}$ yields $\tau'(p) = -\frac{1}{p^2}$. The following calculations can be used to evaluated the expectation in the denominator of the CRLB:

$$\frac{\partial^2}{\partial p^2}\ln f(x;p) = -\frac{1}{p^2} - \frac{x}{(1-p)^2}$$
$$\mathbb{E}\left(\frac{\partial^2}{\partial p^2}\ln f(X;p)\right) = -\frac{1}{p^2} - \frac{E(X)}{(1-p)^2} = -\frac{1}{p^2} - \frac{\frac{1}{p}-1}{(1-p)^2} = -\frac{1}{p^2(1-p)}$$

The CRLB equals

$$\frac{[\tau'(p)]^2}{-n\,\mathbb{E}\left(\frac{\partial^2}{\partial p^2}\ln f(X;p)\right)} = \frac{\frac{1}{p^4}}{\frac{n}{p^2(1-p)}} = \frac{1-p}{np^2}.$$

(d) We compute the mean and variance of $\hat{\theta}$. The expectation is $\mathbb{E}(\hat{\theta}) = \mathbb{E}(\bar{X}) = \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}(X_i) = \frac{1-p}{p} = \theta$. Because X is shifted version of Y, we have $\mathbb{V}\mathrm{ar}(X) = \mathbb{V}\mathrm{ar}(Y) = \frac{1-p}{p^2}$. The variance of $\hat{\theta}$ is therefore

$$\mathbb{V}\mathrm{ar}(\hat{\theta}) = \mathbb{V}\mathrm{ar}(\bar{X}) = \mathbb{V}\mathrm{ar}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{V}\mathrm{ar}(X_i) = \frac{1-p}{np^2}.$$

The estimator $\hat{\theta}$ is unbiased for θ and attains the CRLB. We conclude that $\hat{\theta} = \bar{X}$ is an UMVUE for θ.

(e) $\lim_{n\to\infty} \text{MSE}(\hat{\theta}_n) = \lim_{n\to\infty} \mathbb{V}\text{ar}(\hat{\theta}_n) = \lim_{n\to\infty} \frac{1-p}{np^2} = 0$ which shows that the MLE $\hat{\theta}_n = \bar{X}_n$ of θ is MSE consistent for θ.

(f) Under regularity conditions we know that the asymptotic distribution of MLEs is normal with mean θ and the variance being equal to the CRLB (see page 316 of B&E). Thus, for large n, approximately

$$\hat{\theta} \sim \text{N}\left(\theta, \frac{1-p}{np^2}\right).$$

It is mathematically neater to write $\sqrt{n}\big(\hat{\theta} - \theta\big) \xrightarrow{d} \text{N}\left(0, \frac{1-p}{p^2}\right)$.